COMPLEMENTARY SET LOGIC

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June 2023/26

Hierarchies in Peano's world

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Content

1 Explicitly defined logical language

Giuseppe Peano was the first mathematician to publish rigorously formalized logical calculi in a readily verbalizable symbolic language, abbreviating words by their initials [6][7]. However, due to alleged antinomies, his logical calculi failed to gain acceptance. In fact, they are consistent and highly efficient. This has been demonstrated in their update in the Peano chapters of the *Universallogik* [4]. His logic in the tradition of Aristotle, Leibniz and Boole has a natural and very simple syntax for terms including propositions. Five **basic terms** suffice as a logical foundation, namely the identity, the conjunction, the negation, the universe and class terms, each with a defined verbalization:

Definitions:

A IS IDENTICAL TO B $:=$ $(A{=}B)$	*identity*
A AND B $:=$ $A{\cdot}B$	*conjunction*
NOT A $:=$ $\neg A$	*negation*
EXISTENT $:=$ I	*universe*
x WITH f_x $:=$ $\{x\vert f_x\}$	*class term*

Free variables, replaceable by any terms, are always written in capital letters.
Bound variables are always written in lower case letters like x in f_x .

These basic terms create a exeptionally powerful logical language, exclusively using **standard definitions** $a{:=}b$, which explain new terms a by already known terms b. The entire mathematical language is defined in this explicit form, as Peano's logic includes set theory and thus the current foundation of mathematics. Specifically, Neumann's set hierarchy, which until now has only been defined implicitly through recursive axioms, is also explicitly defined here. It is well known that it encompasses all sets of Zermelo-Fraenkel set theory. However, Peano's complementary set logic contains significantly more classes. Here, in addition to Neumann's hierarchy, manifold other hierarchies can be defined in an analogous manner.

The structure of the logical language is discussed in the following sections. Each section presents a logical level of the *Universallogik*, each with a few short axioms. In total, twelve axioms are sufficient to prove all mathematical theorems. Only theorems later used to describe hierarchies are quoted from the *Universallogik*, with the page of the proof given in the margin as [...]. Sources for historical remarks can also be found in this book.

2 Logic

Aristotle viewed logic as proof science. He introduced rules of inference for the first time. They are updated here in a precise form and notated with derivation operators $\vdash$ and $\dashv\vdash$, which do not occur in logical texts. Their metavariables for terms have serifs in contrast to logical variables.

$a \vdash b$ means: If (every) a is provable, then (every) b is provable as well.
$a \dashv\vdash b$ stands for the rules $a \vdash b$ and $b \vdash a$.

Two rules of inference of Aristotle's are basic. One is known as modus ponens and uses an implication, however, it is not the usual implication in today's propositional calculi, but an algebraic implication that Leibniz first defined precisely:

Definition:
$(A{\Rightarrow}B) := (A{=}A{\cdot}B)$

Axioms:

$a, a{\Rightarrow}b \vdash b$	*modus ponens*	A1	[38]
$a \vdash a{=}\mathsf{I}$	*judgment*	A2	[36]

Elementary logic also includes an axiom of identity and three axioms for the conjunction, describing a multiplicative idempotent monoid corresponding to the composition of terms in natural language:

Definitions:
$A{\cdot}B{\cdot}C := (A{\cdot}B){\cdot}C$
$A{\cdot}B{\cdot}C{\cdot}D := (A{\cdot}B{\cdot}C){\cdot}D$ etc.
$A_1 \ldots A_n\ N := (A_1{\cdot} \ldots {\cdot}A_n{\cdot}N)$ for defined adjectives $A_1 \ldots A_n$ and nouns N

Axioms:

$(A{=}B) \Rightarrow (xAy{\Rightarrow}xBy)$	*identity axiom*	A3	[38]
$\mathsf{I}{\cdot}A = A$	*neutral*	A4	[36]
$A{\cdot}A = A$	*idempotent*	A5	[36]
$A{\cdot}B{\cdot}C = A{\cdot}C{\cdot}B$	*permutative*	A6	[36]

These six simple logical axioms suffice to derive all important proof procedures, because rules of inference, equations, and implications are essential jusitifications in proofs. The proof technique is described in a user-friendly manner on four pages in the universal logic [115-118]. It allows precise calculus proofs that fully correspond to the usual more informal proofs. All calculus proofs there and also here in this paper are linear deductions as a sequence of terms between which the justifications are inserted as indices.

The following explanations of justifications may suffice here:

- Axioms and theorems are used with their $_{name}$, definitions, recognizable by the character eliminated or introduced, with $_{def}$.
- Hypotheses of the theorem proved are labeled with $_{hyp,\ hyp1,\ hyp2}$.
- Single factors of proof terms are valid since the product is a conjunction; this also applies for the domain of a quantifier.
- Underlined factors are referenced in sequential order with $_{U,\ U1,\ U2,\ \dots}$.
- $_{-name}$ indicates the omission of a factor.
- $_{+name}$ indicates the insertion of a factor.
- $_{>name}$ indicates the application of a factor to an term named as $_{name}$.
- $_{B}$ indicates automatic proof steps in Boolean Algebra, which are not carried out here.

3 Classical logic

Classical logic also uses negation. As is well known, all other operators of classical logic can be defined with negation and conjunction. All can be used for terms and propositions, but different symbols and verbalizations are usually chosen:

Definitions for terms and sets:	for propositions:
ENTITY := THING := I	YES := I
0 := ¬I	NO := 0
A CUT WITH B := $A∩B$:= $A·B$	$A∧B$:= $A·B$
A UNITED WITH B := $A∪B$:= ¬(¬A·¬B)	A OR B := $A∨B$:= ¬(¬A·¬B)
A WITHOUT B := $A\setminus B$:= A·¬B	
A IS INCLUDED IN B := $A⊆B$:=(A=$A·B$)	A IMPLIES B:=($A⇒B$):=(A=$A·B$)
ALL A ARE B := (A=$A·B$)	
A IS DIFFERENT FROM B := $A≠B$:= ¬(A=B)	
$A⊂B$:= ($A⊆B$)·($A≠B$)	

Leibniz characterized negation effective and elegant by a synonym of implication added to the logic axioms:

Axiom:
(A=$A·B$) = (A·¬B=0) *negation rule* A7 [49]

From the axioms A1-A7 follow the rules of classical propositional calculi [2] and Boolean algebra. So, like Boole, one can do automatic proofs for any classical theorem about terms or propositions [127]. Included are *indirect* proofs, proofs with *contraposition*, *ex falso quodlibet*, *reflexive* or other known rules not quoted here.

4 Identity logic with propositional sublogic

Logicians up to Boole used equations and implications only as rules, not yet as full terms that can be substituted into all variables. Leibniz was the first to do this. Two rules of his guarantee an effective identity logic:

Axioms:

$(A{\neq}B) = ((A{=}B){=}0)$	*incoincidence*	A8	[49]
$(A{\Rightarrow}B){\cdot}A \Rightarrow B$	*substitution*	A9	[49]

Theorems:

$(A{=}B) \Rightarrow (xAy{=}xBy)$	*congruent*	[128]
$(A{=}B){\cdot}xAy = (A{=}B){\cdot}xBy$	*change*	[128]
$(A{\vee}B){\cdot}(B{\Rightarrow}C) \Rightarrow A{\vee}C$	*disjunctive*	[129]
$A{\subseteq}A$	*reflexive*	[130]
$(A{\subseteq}B){\cdot}(B{\subseteq}A){=}(A{=}C)$	*antisymmetric*	[131]
$(A{\subseteq}B){\cdot}(B{\subseteq}C){\Rightarrow}(A{\subseteq}C)$	*transitive*	[130]
$(A{\subset}B){\cdot}(B{\subset}C){\Rightarrow}(A{\subset}C)$	*transitive*	[131]
$(A{\subset}B){\cdot}(B{\subseteq}C){\Rightarrow}(A{\subset}C)$	*transitive*	[131]
$0{\subseteq}A \quad A{\subseteq}\mathsf{I}$	*extreme*	[131]
$A{\cdot}B{\subseteq}A, \quad A{\subseteq}A{\cup}B$	*upper term*	[130f]
$(A{\subseteq}C){\cdot}(B{\subseteq}C) = (A{\cup}B){\subseteq}C$	*subsum*	[131]

As Leibniz didn't publish anything on logic, his brilliant ideas remained fruitless, especially his intelligent embedding of propositional logic in term logic by the proposition rule $p{=}(p{=}\mathsf{I})$. Syntax constructions for higher logics are thus superfluous. The logic automatically includes a two-valued propositional sublogic generated by equations [50].

Theorems:

$p = (p{=}\mathsf{I})$	for propositions p	*true*	[212]
$\neg p = (p{=}0)$	for propositions p	*false*	[212]

As the term logic allows a conjunction of propositions with terms, conditional definitions with case distinctions, which in conventional calculi cannot be defined by a single equation, can also be formulated explicitly:

Definition:

$$A \text{ IF } X \text{ ELSE } B := A{\cdot}X{\vee}B{\cdot}\neg X$$

Theorem for definienda δ, terms a, b and propositions p:

$\delta := a \text{ IF } p \text{ ELSE } b \;\dashv\vdash\; p{\Rightarrow}(\delta{=}a), \neg p{\Rightarrow}(\delta{=}b)$ — *conditional definition* [132]

5 Class logic

Leibniz formed classes by uniting individuals in the sense of one-element classes. Peano defined them in his calculus of 1889/90, which is characterized by his rule for the content of classes and an individual axiom from Leibniz:

Definitions:

$\textsf{EQUAL}\ A := \{A\} := \{x|x{=}A\}$

$\{A, B\} := \{A\}\cup\{B\}$

$\{v_1, ..., v_n\} := \{v_1, ..., v_{n-1}\}\cup\{v_n\}$ for $n>2$ and variables v_1, ..., v_n

A IS IN B := A IS A B := A IS B := $A{\in}B := \{A\}{\cdot}B{\neq}0$

$\{x{\in}A|f_x\} := A{\cdot}\{x|f_x\}$

THE POWER OF A := CLASS OF A := $\mathcal{P}A := \{x|x{\subseteq}A\}$

Axioms:

$A{\in}B \Rightarrow \{A\}{\subseteq}B$	*individual*	A10	[55]
$X{\in}s{=}f_X{\cdot}(X{\in}\mathrm{I}) \dashv\vdash s{=}\{x|f_x\}$	*content*	A11	[73]

Theorems:

$A{\in}\mathrm{I} = \{A\}{\neq}0 = A{\in}\{A\}$		*existent*	[132]
$A{\notin}0$		*empty*	[132]
$X{\in}A{\Rightarrow}A{\neq}0$		*not empty*	[132]
$A{\in}B{\Rightarrow}A{\in}\mathrm{I}$	$\{x|f_x\}{=}\{x|f_x{\cdot}(x{\in}\mathrm{I})\}$	*real*	[132]
$(A{\in}B){\cdot}(B{\subseteq}C){\Rightarrow}A{\in}C$		*syllogism*	[133]
$X{\in}A{\Rightarrow}X{\in}(A{\cup}B)$		*extensive*	[133]
$X{\in}(A{\cdot}B){=}(X{\in}A){\cdot}(X{\in}B)$		*distributive*	[133]
$X{\in}(A{\cup}B){=}(X{\in}A){\vee}(X{\in}B)$		*distributive*	[133]
$A{\in}\neg B{=}(A{\in}\mathrm{I}){\cdot}(A{\notin}B)$	$X{\in}(A{\setminus}B){=}(X{\in}A){\cdot}(A{\notin}B)$	*exept*	[133]
$A{\in}\mathrm{I} \Rightarrow (A{\in}\{B\}{=}(A{=}B))$		*equal*	[134]
$C{\in}X \Rightarrow C{\in}\{v_1, ..., v_n\}{=}(C{=}v_1){\vee}...{\vee}(C{=}v_n)$		*ambiguous*	[134]
$v_i{\in}X \Rightarrow v_i{\in}\{v_1, ..., v_n\}$	for integers $n{\geq}i{\geq}1$	*listed*	[134]
$X{\in}\{x|f_x\}{=}(X{\in}\mathrm{I}){\cdot}f_X$	$X{\in}\{x{\in}A|f_x\}{=}(X{\in}A){\cdot}f_X$	*satisfied*	[140]
$X{\in}a{=}X{\in}b \vdash a{=}b$	$X{\in}a{\Rightarrow}X{\in}b \vdash a{\subseteq}b$	*free*	[140]
$\{x|a{\cdot}p\}{=}p{\cdot}\{x|a\}$ $\{x|p\}{=}p$ for x-free propositions p		*unbound*	[140]
$\mathcal{P}0{=}\{0\}$		*zero power*	[143]
$\mathcal{P}\{A\}{=}\{0, \{A\}\}$		*unit power*	[143]
$\mathcal{P}\{A, B\}{=}\{0, \{A\}, \{B\}, \{A, B\}\}$		*pair power*	
$X{\subseteq}Y{\Rightarrow}\mathcal{P}X{\subseteq}\mathcal{P}Y$		*monotonic*	[143]

Bound variables always refer to existing terms with $x{\in}\mathrm{I}$; only they can be elements, but never non-existent terms with $x{\notin}\mathrm{I}$. Existence conditions $x{\in}\mathrm{I}$ are therefore negligible for bound variables. In the case of free variables, however, the existence or non-existence must always be taken into account. Because class logic has no existence axioms, proofs in class calculus are more elementary than in set theory. Therefore proofs of theorems are given here, even if they seem trivial or known:

Proof of *pair power*:

$_{free}$: $X{\in}\mathcal{P}\{A,B\}$ $_{satisfied}$ $(\underline{X{\in}\mathrm{I}})\cdot(\underline{X{\subseteq}\{A,B\}})$ $_{def,\ B}$ $\underline{X{=}X\cdot\{A\}{\cup}X\cdot\{B\}}$;
case 1 $X{=}0$: $X{=}0$ $_{>U1\ listed}$ $X{\in}\{0,\{A\},\{B\},\{A,B\}\}$;
case 2 $A{\in}X$, *case 3* $B{\in}X$: $(A{\in}X)\cdot(B{\in}X)$ $_{individual}$ $(\{A\}{\subseteq}X)\cdot(\{B\}{\subseteq}X)$ $_{subsum}$
$\{A\}{\cup}\{B\}{\subseteq}X$ $_{def\ +U2\ antisymmetric}$ $\{A,B\}{=}X$ $_{>U1,\ listed}$ $X{\in}\{0,\{A\},\{B\},\{A,B\}\}$;
case ¬*1* $X{\neq}0$: *case* ¬*2 (similar case* ¬*3)*: $A{\notin}X$ $_{def\ B}$ $\{A\}\cdot X{=}0$
$_{>U3\ B}$ $X{=}X\cdot\{B\}$ $_{def,\ >case\ \neg 1}$ $(\underline{X{\subseteq}\{B\}})\cdot(X\cdot\{B\}{\neq}0)$ $_{B\ def}$ $B{\in}X$ $_{individual}$
$\{B\}{\subseteq}X$ $_{+U4\ antisymmetric}$ $\{B\}{=}X$ $_{>U1,\ listed}$ $X{\in}\{0,\{A\},\{B\},\{A,B\}\}$;
$_{reverse,\ free}$: $X{\in}\{0,\{A\},\{B\},\{A,B\}\}$ $_{real\ (def)\ ambiguous}$
$(\underline{X{\in}\mathrm{I}})\cdot((X{=}0)\lor(X{=}\{A\})\lor(X{=}\{B\})\lor(X{=}\{A,B\}))$ $_{>extreme\ >upper\ term\ >reflexive}$
$_{(disjunctive)}$ $(X{\subseteq}\{A,B\})\lor(X{\subseteq}\{A\}{\cup}\{B\})\lor(X{\subseteq}\{A\}{\cup}\{B\})\lor(X{\subseteq}\{A,B\})$
$_{def,\ B}$ $X{\subseteq}\{A,B\}$ $_{+U5\ satisfied}$ $X{\in}\mathcal{P}\{A,B\}$ ■

Of course any classes can be defined, whether they exist or not. For example Russell's class, the prime example of a non-existent class, is a well defined term, useful to define conditional terms:

Definitions:

A IS NO B := $A{\notin}B$:= $\neg(A{\in}B)$
THE RUSSELL CLASS := † := $\{x|x{\notin}x\}$
A IF B := A IF B ELSE †

6 Defined predicate logic

In 1888, Peano introduced **explicitly defined quantifiers** in his class logic. No later logician used his idea, although it is ingenious and produces without additional axioms an efficient multi-sort predicate logic as part of propositional logic, because relativized quantifiers for any terms can also be explicitly defined:

Definitions:

f_x FOR ALL x := $\forall x{:}f_x$:= $(\{x|f_x\}{=}\mathrm{I})$
f_x FOR ALL x IN A := $\forall x{\in}A{:}f_x$:= $\forall x{:}(x{\in}A{\Rightarrow}f_x)$
$f_{x,y}$ FOR ALL x,y IN A := $\forall x,y{\in}A{:}f_{x,y}$:= $\forall x{:}\forall y{\in}A{:}(x{\in}A{\Rightarrow}f_{x,y})$
EXISTS x WITH f_x := $\exists x{:}f_x$:= $(\{x|f_x\}{\neq}0)$
EXISTS x IN A WITH f_x := $\exists x{\in}A{:}f_x$:= $\exists x{:}((x{\in}A)\cdot f_x)$

Theorems for propositions f_X and terms a:

$X{\in}a{\Rightarrow}f_X \vdash \forall x{:}f_x, \forall x{\in}a{:}f_x$	*general*	[138]
$(X{\in}a){\cdot}p{\Rightarrow}f_X \vdash p{\Rightarrow}\forall x{\in}a{:}f_x$	*general*	[138]
$(X{\in}A){\cdot}\forall x{:}f_x{\Rightarrow}f_X,\ (X{\in}A){\cdot}\forall x{\in}A{:}f_x{\Rightarrow}f_X$	*special*	[138]
$(X{\in}A){\cdot}f_X{\Rightarrow}\exists x{:}f_x{\cdot}\exists x{\in}A{:}f_x$	*quantify*	[138]
$\forall x{\in}A{:}f_x{\cdot}\forall x{\in}A{:}h_x = \forall x{\in}A{:}(f_x{\cdot}h_x)$	*quantifier junction*	[139]
$A{\subseteq}B = \forall x{:}(x{\in}A{\Rightarrow}x{\in}B) = \forall x{\in}A{:}(x{\in}B)$	*subset synonym*	[141]
$\exists x{:}(x{\in}A){=}(A{\neq}0)$	*not empty*	[141]
$\exists x{:}(f_x{\cdot}p) = p{\cdot}\exists x{:}f_x$ for x-free propositions p	*not quantified*	[143]

With quantifiers you can define many other classes, for example union or classes whose members are described by bound variable terms that require existential quantifiers for precise definitions:

Definitions:

$\{T_x|f_x\} := \{y|\exists x{:}(f_x{\cdot}(y{=}T_x))\}$

$\{T_{x,y}|f_{x,y}\} := \{z|\exists x{:}\exists y{:}(f_{x,y}{\cdot}(z{=}T_{x,y}))\}$

THE UNION OF ALL $A := \bigcup A := \{x|\exists y{:}((y{\in}A){\cdot}(x{\in}y))\}$

Theorems:

$T_X{\in}\{T_x|f_x\}{=}(T_X{\in}\mathrm{I}){\cdot}f_X$	*satisfied*	[143]
$\bigcup 0{=}0$	*zero union*	[144]
$A{\in}\mathrm{I} \Rightarrow \bigcup\{A\}{=}A$	*unit union*	[144]
$A{\subseteq}B \Rightarrow \bigcup A{\subseteq}\bigcup B$	*monoton*	[144]
$\bigcup(A{\cup}B) = \bigcup A{\cup}\bigcup B$	*union of a sum*	[144]
$Y{\in}X \Rightarrow Y{\subseteq}\bigcup X$	*upper bound*	[144]
$Y{\in}X{\Rightarrow}(A{\in}\mathcal{P}Y{\Rightarrow}A{\in}\mathcal{P}\bigcup X)$	*upper power*	

Proof of *upper power*:

hyp2-satisfied +hyp1-upper bound $(A{\subseteq}Y){\cdot}(Y{\subseteq}\bigcup X)$ *transitive* $A{\subseteq}\bigcup X$ *satisfied (hyp2-real)* $A{\in}\mathcal{P}\bigcup X$ ∎

7 Defined relation logic with functions

In 1888 Peano introduced ordered pairs for relations. In 1914, Hausdorff defined functions as univalent relations. Explicit definitions need Peano's operator $\imath$ that selects the member of a one-element class and Kuratowski's pair from 1921. They generate the complete terminology of functions in a direct and general way for any classes:

Definitions:

(A,B) := $\{\{A,B\},\{A\}\}$ IF $(A{\in}\mathsf{I})\cdot(B{\in}\mathsf{I})$

THE A := ιA := $\bigcup A$ IF $\exists x{:}(A{=}\{x\})$

THE F OF X := $F(X)$:= $\iota\{y|(X,y){\in}F\}$

THE FUNCTION F ON A := $F{\restriction}_A$:= $\{(x,y)|(x{\in}A)\cdot(y{=}F(x))\}$

THE IMAGE OF A UNDER F := $F[A]$:= $\{F(x)|x{\in}A\}$

VALUE OF F := $\mathrm{Im}F$:= $F[\mathsf{I}]$

ARGUMENT OF F := $\mathrm{Def}F$:= $\{x|F(x){\in}\mathsf{I}\}$

MAP := MAPPING := $\{f|f{=}f{\restriction}_{\mathsf{I}}\}$

MAPPING FROM A TO B := $A{\longrightarrow}B$:= $\{f{\in}\text{MAP}|(A{=}\mathrm{Def}f)\cdot(f[A]{\subseteq}B)\}$

$F{:}A{\longrightarrow}B$:= $F{\in}(A{\longrightarrow}B)$

IDENTITY := Id := $\{(x,y)|x{=}y\}$

$G{\circ}F$:= $\{(x,y)|y{=}G(F(x))\}$

Theorems (with the axiom $\{x,y\}{\in}\mathsf{I}$, provable in set logic):

$F(X){\in}\mathsf{I}{=}X{\in}\mathrm{Def}F$	*argument*	[145]
$X{\in}A \Rightarrow (F(X){=}F{\restriction}_A(X))$	*equal value*	[146]
$\mathrm{Def}(F{\restriction}_A) = A\cdot\mathrm{Def}F$, $\mathrm{Im}(F{\restriction}_A) = F[A]$	*restriction*	[146]
$A{\subseteq}\mathrm{Def}F = (A{=}\mathrm{Def}(F{\restriction}_A))$	*subfunction*	
$(F{\restriction}_A){\restriction}_B = F{\restriction}_{A\cdot B}$	*double restriction*	[146]
$F{\restriction}_0{=}0$, $F[0]{=}0$	*empty function*	[147]
$F[\{A\}] = \{F(A)\}$	*single image*	[147]
$F[A{\cup}B] = F[A]{\cup}F[B]$	*sum image*	[147]
$A{\subseteq}B \Rightarrow F[A]{\subseteq}F[B]$	*monotonic*	[147]
δ := $\{(x,y)|y{=}\varphi_x\}$	*graph*	[148]
$\vdash A{\in}\mathrm{Def}\delta{\Rightarrow}(\delta(A){=}\varphi_A)$, $\mathrm{Def}\delta{=}\{x|\varphi_x{\in}\mathsf{I}\}$		
$\mathrm{Id}[A]{=}A$	*identical image*	[149]
$(G{\circ}F)[A] = G[F[A]]$	*image of image*	[149]
$Y{\in}\bigcup F[X]{=}\exists z{\in}X{:}((Y{\in}F(z))\cdot(F(z){\in}\mathsf{I}))$	*value element*	

Proofs:

subfunction:

$A{\subseteq}\mathrm{Def}F$ ${}_{\textit{def}}$ $A{=}A\cdot\mathrm{Def}F$ ${}_{\textit{restriction}}$ $A{=}\mathrm{Def}(F{\restriction}_A)$ ■

empty function (first formula proved in [147]):

$F[0]$ ${}_{\textit{def}}$ $\{y|\exists y{:}((y{=}F(x))\cdot(x{\in}0))\}$ ${}_{\textit{empty-false}}$ $\{y|\exists y{:}((y{=}F(x))\cdot 0)\}$

${}_{\textit{not quantified, unbound}}$ $0\cdot\{y|\exists y{:}(y{=}F(x)\}$ ${}_B$ 0 ■

value element:

$Y{\in}\bigcup F[X]$ ${}_{\textit{satisfied (2x)}}$ $\exists a{:}(\exists z{:}((a{\in}\mathsf{I})\cdot(a{=}F(z))\cdot(z{\in}X))\cdot(Y{\in}a))$ ${}_{\textit{not quantified}}$

$\exists a{:}\exists z{:}((Y{\in}a)\cdot(a{\in}\mathsf{I})\cdot(a{=}F(z))\cdot(z{\in}X))$ ${}_{\textit{change, not quantified}}$

$\exists z{:}((Y{\in}F(z)){\cdot}(F(z){\in}\mathsf{I}){\cdot}(z{\in}X))$ ${}_{\textit{def (B)}}$ $\exists z{\in}X{:}((Y{\in}F(z)){\cdot}(F(z){\in}\mathsf{I}))$
${}_{\textit{reverted: satisfied}}$ $\exists z{:}((Y{\in}F(z)){\cdot}(F(z){\in}F[X]))$ ${}_{\textit{quantify}}$ $\exists z{:}\exists a{:}((Y{\in}a){\cdot}(a{\in}F[X]))$
${}_{\textit{+real not quantified}}$ $(Y{\in}\mathsf{I}){\cdot}\exists a{:}((Y{\in}a){\cdot}(a{\in}F[X]))$ ${}_{\textit{satisfied}}$ $Y{\in}\bigcup F[X]$ ■

8 Defined set logic

Leibniz used an verbal **binary set logic** in his *Monadology* of 1714 [1]. **Peano** published1897 the first formalized set axioms in his *Logique mathématique* [7], namely $(A{\in}\mathbb{K}){\cdot}(B{\in}\mathbb{K}){\Rightarrow}A{\cdot}B{\in}\mathbb{K}$, $A{\in}\mathbb{K}{\Rightarrow}\neg A{\in}\mathbb{K}$, $\{A\}{\in}\mathbb{K}$; they generate a complementary lattice $\mathbb{K}$. Russell and Zermelo thought that the Russell class also existed in $\mathbb{K}$, and therefore called his logic inconsistent [74]. However, models show that Peano's **complementary set logic** is consistent [83f]. His universe $\mathbb{K}$ can be defined explicitly, so that his axioms are provable [154]. The universe of Leibniz $\mathbb{L}$ and a smaller extensional universe $\mathbb{E}$ can be defined too. This is done here in the easily understandable *Verbal Logic* [5], which defines a precise term language at a current level. Short formulas for proofs result by definition.

Definitions:

SINGLE := x WITH $\{x\}$ IS EXISTENT
COMBINABLE := x WITH $x{\cdot}y$ IS EXISTENT FOR ALL y
NEGATABLE := x WITH NOT x IS EXISTENT
IDEA := x WITH (x IS A SET = x IS A CLASS OF SETS)
$\mathbb{K}$:= COMBINABLE NEGATABLE SINGLE IDEA

TRUE := EQUAL YES
FALSE := EQUAL NO
EXTENSIONAL := x WITH $\{x\}$ IS EXISTENT AND $x{\cup}y$ IS EXISTENT FOR ALL y
BINARY := x WITH $x{=}x$ IS TRUE AND $x{\neq}x$ IS FALSE
$\mathbb{L}$:= BINARY EXTENSIONAL IDEA

FALSIFIABLE := x WITH $x{\cdot}\neg x$ IS FALSE
$\mathbb{E}$:= FALSIFIABLE EXTENSIONAL IDEA

Theorems:

BINARY = $(\mathsf{I}{\in}\{\mathsf{I}\}){\cdot}(0{\in}\{0\}) = (\mathsf{I}{\in}\mathsf{I}){\cdot}(0{\in}\mathsf{I})$	*binary synonym*	
FALSIFIABLE = $0{\in}\{0\} = 0{\in}\mathsf{I}$	*falsifiability*	
$\mathbb{K} \vdash (0{\in}\mathsf{I}){\cdot}(\mathsf{I}{\in}\mathsf{I})$	*bit exists*	[153]
$\mathbb{K} \vdash A{\in}\mathsf{I}{\Rightarrow}\forall y{:}(A{\cup}y{\in}\mathsf{I}), \{A\}{\in}\mathsf{I}$	*extensional*	[154]
$\mathbb{K} \vdash \mathbb{L} \vdash \mathbb{E}$	*chooseable universe*	

Proofs:

binary synonym:

BINARY ${}_{def}$ $\{x|((x{=}x)\in\{\mathsf{I}\})\cdot((x{\neq}x)\in\{0\})\}$ ${}_{reflexive\text{-}true/false}$ $\{x|(\mathsf{I}\in\{\mathsf{I}\})\cdot(0\in\{0\})\}$ ${}_{unbound}$ $(\mathsf{I}\in\{\mathsf{I}\})\cdot(0\in\{0\})$ ${}_{existent}$ $(\mathsf{I}\in\mathsf{I})\cdot(0\in\mathsf{I})$;

falsifiability:

$\mathsf{FALSIFIABLE}$ ${}_{def}$ $\{x|(x\cdot\neg x)\in\{0\}\}$ ${}_{B,\ unbound}$ $0\in\{0\}$ ${}_{existent}$ $0\in\mathsf{I}$ ■

chooseable universe:

$\mathbb{K}$ ${}_{extensional1\ general}$ $\forall x\in\mathsf{I}:\forall y:(x\cup y\in\mathsf{I})$ ${}_{+extensional2}$ $\forall x\in\mathsf{I}:((\{x\}\in\mathsf{I})\cdot\forall y:(x\cup y\in\mathsf{I}))$ ${}_{def,\ B}$ $\underline{\mathsf{EXTENSIONAL}{=}\mathsf{I}}$; $\mathbb{K}$ ${}_{def\text{-}factors}$ IDEA ${}_{B}$ $\mathsf{I}\cdot\mathsf{IDEA}$ ${}_{U\ +bit\ exists}$ $(0\in\mathsf{I})\cdot(\mathsf{I}\in\mathsf{I})\cdot\mathsf{EXTENSIONAL\ IDEA}$ ${}_{binary\ synonym\ def}$ $\mathbb{L}$ ${}_{def\text{-}factor,falsifiability\ def}$ $\mathbb{E}$ ■

An **explicit set definition** determines the subclass $\mathbb{M}$ of all sets in the universe and permits a **derivable Cantor set theory** in which complements of sets are not sets and urelements in the sense of his 'objects of our intuition' are possible [95]:

Definitions:

$\mathsf{URELEMENT} := \mathbb{U} := \{x|\neg\exists y\in x:(y\neq x)\}$

$\mathsf{FOUNDED} := \mathbb{F} := \{x|\forall y\in\mathcal{P}x\backslash\mathbb{U}:((\Cap y\in y)\cdot(y\cdot\Cap y=0))\}$

$\mathsf{POWERFULL} := x$ WITH THE POWER OF x IS EXISTENT

$\mathsf{SUMMARIZING} := x$ WITH THE UNION OF ALL x IS EXISTENT

$\mathbb{M}$:= MENGE := SET := POWERFULL SUMMARIZING FOUNDED THING

Theorems:

$\mathsf{URELEMENT}{=}\{x|(x{=}0)\cdot(x{=}\{x\})\}$	*urelements*	[180]
ALL URELEMENTS ARE SETS, $\mathbb{U}\subseteq\mathbb{M}$	*primitive sets*	[180]

In $\mathbb{K}$, $\mathbb{L}$ or $\mathbb{E}$, set theory follows from Cantor's axiom that set images exist [110]. These are handled like sets according to analogous rules (*):

Axiom:

$A\in\mathbb{M}\Rightarrow F[A]\in\mathsf{I}$	*image of sets* A12	[96]

Definition:

$\mathsf{IMAGE} := \mathsf{BILD} := \mathbb{B} := \{f[x]|x\in\mathbb{M}\}$

Theorems:

$\{A,B\}\in\mathbb{B},\ \{A\}\in\mathbb{B},\ 0\in\mathbb{B},\ \{A\}\in\mathsf{I},\ 0\in\mathsf{I}$	* *elemental*	[178]
$A\in\mathbb{B}\Rightarrow G[A]\in\mathbb{B}$	* *real image*	[177]
$(A\subseteq C)\cdot(C\in\mathbb{B})\Rightarrow A\in\mathbb{B}$	* *real part*	[178]
$(A\in\mathbb{B})\cdot(A\subseteq\mathbb{B})\Rightarrow\bigcup A\in\mathbb{B}$	* *real union*	[180/5]
$A\in\mathbb{B}\Rightarrow\mathcal{P}A\in\mathbb{B}$	* *real power*	[178]

$A\in\mathbb{M}\Rightarrow A\subseteq\mathbb{M}$	*sets of sets*	[154]
$A\in\mathbb{M}\Rightarrow\neg A\notin\mathbb{M},\quad \mathsf{I}\notin\mathbb{M}$	*not-sets*	[164f]
$A\in\mathbb{M}\Rightarrow F\restriction_{A}\in\mathsf{MAP}$	*submap*	[164]
$A\in\mathbb{M}\Rightarrow A\in\mathbb{B},\ \mathbb{M}\subseteq\mathbb{B}$	*subclass*	[175ff]
$A\in\mathbb{B}\Rightarrow\mathcal{P}A\neq F[A]$	*no image*	[*164]

Proof of *elemental*:

proved [178]: $\{A,A\}\in\mathbb{B}$ *def B* $\{A\}\in\mathbb{B}$ *+extreme* $(0\subseteq\{A\})\cdot(\{A\}\in\mathbb{B})$ *real part* $0\in\mathbb{B}$ *real* ■

Sequences are usually formed by counting with ordinal numbers. Cantor was the first do extend this to the transfinite. In 1915, Zermelo first precisely defined ordinal numbers with a counting operator A' and the reverse count $\bigcup A$; his elegant implicit definition can be converted into a short **explicit ordinal number definition** [98]:

Definitions:

$\text{THE NUMBER AFTER } A := A' := A\cup\{A\}$

$1:=0' \quad 2:=1' \quad 3:=2'$

$\text{ORDINAL} := \mathbb{O} := \{z\in\mathbb{M}\,|\,\forall x\in z{:}(x'\in z')\cdot\forall y\in\mathcal{P}z{:}(\bigcup y\in z')\}$

$\text{SEQUENCE} := \text{SEQ} := \{f\,|\,\exists z\in\mathbb{O}{:}(f{:}z\longrightarrow\mathsf{I})\}$

Theorems:

$(X\in\mathbb{O})\cdot(Y\in\mathbb{O})\Rightarrow(X\subset Y)=(X\in Y)$	*ordered*	[166]
$Z\in\mathbb{O}\Rightarrow Z\subseteq\mathbb{O}$	*number set*	[166]
$(Z\in X)\cdot(X\in\mathbb{O})\Rightarrow(Z\in\mathbb{O})\cdot(Z\neq X)\cdot(Z\subseteq X)\cdot(Z=Z\cdot X)$	*subnumber*	
$(A\in\mathbb{O})\cdot(B\in\mathbb{O})\Rightarrow(A\subset B)\vee(B\subseteq A)$	*linear*	[166]
$\mathbb{O}\notin\mathsf{I},\quad\mathbb{O}\notin\mathbb{M}$	*totality*	[167]
$X\in Y\Rightarrow(X\in A'=(X\in A)\vee(X=A))$	*smaller*	[167]
$A\in X\Rightarrow A\in A'$	*count in*	[167]
$A\in\mathbb{O}\Rightarrow A'\in\mathbb{O}$	*count*	[168]
$0\in\mathbb{O},\ 1\in\mathbb{O},\ 2\in\mathbb{O},\ 3\in\mathbb{O}$	*ordinal*	[169]
$(X\in\mathbb{O})\cdot(X\neq 0)\cdot(X\neq 1)\Rightarrow 1\in X$	*ordinal above 1*	
$A\in\mathbb{O}\Rightarrow(\bigcup A\in\mathbb{O})\cdot(\bigcup(A')=A)$	*countdown*	[168]
$A\in\mathbb{O}\Rightarrow(A=\bigcup A)\vee(A=(\bigcup A)')$	*countdown limit*	[168]
$(Z\in X)\cdot(X\in\mathbb{O})\cdot(X=\bigcup X)\Rightarrow Z'\in X$	*count limit*	
$(Z\in\mathbb{O})\cdot(Z\neq\bigcup Z)\Rightarrow\bigcup Z\in Z$	*predecessor*	
$A\in\mathbb{O}\Rightarrow A'\neq\bigcup(A')$	*successor*	
$(a\subseteq\mathbb{O})\cdot(0\in a)\cdot(a\subseteq\mathcal{P}a)\vdash$	*induction*	[169]
$\quad(0\in K)\cdot\forall x\in a\cdot K{:}(x'\in K)\cdot\forall x\in a{:}((x=\bigcup x)\cdot(x\cdot a\subseteq K)\Rightarrow x\in K)\Rightarrow a\subseteq K$		
$f_0,\ X\in\mathbb{O}\Rightarrow(f_X\Rightarrow f_{X'})\cdot((X=\bigcup X)\cdot\forall z\in X{:}f_z\Rightarrow f_X)$	*transfinite induction*	
$\quad\vdash\forall x\in\mathbb{O}{:}f_x,\quad X\in\mathbb{O}\Rightarrow f_X$		

$\forall x{\in}\mathbb{O}{:}(x{\subseteq}K{\Rightarrow}x{\in}K){\Rightarrow}\mathbb{O}{\subseteq}K$ *transfinite class* [169]

$F{\in}\mathsf{SEQ} \Rightarrow \mathsf{Def}F{\in}\mathbb{O}$ *number arguments* [171]

$(F{\in}\mathsf{SEQ}){\cdot}(Y{\in}\mathsf{I}){\cdot}(G{=}F{\cup}\{(\mathsf{Def}F, Y)\}) \Rightarrow$ *continuation* [172]

$\quad (G{:}\mathsf{Def}F'{\longrightarrow}\mathsf{I}){\cdot}(G{\in}\mathsf{SEQ}){\cdot}\forall x{\in}\mathsf{Def}F{:}(F(x){=}G(x)){\cdot}(Y{=}G(\mathsf{Def}F))$

Proofs:

subnumber:

$_{\textit{hyp2-number set}}\ X{\subseteq}\mathbb{O}\ _{\textit{+hyp1 syllogism}}\ Z{\in}\mathbb{O}\ _{\textit{ordered (hyp1+2)}}\ Z{\subset}X\ _{\textit{def}}\ (Z{\neq}X){\cdot}(Z{\subseteq}X)\ _{\textit{def}}$
$Z{=}Z{\cdot}X$ ■

ordinal above 1: $_{\textit{linear def (hyp1)}}\ (X{\subset}1){\vee}(1{\subset}X){\cdot}(X{\neq}1)\ _{\textit{-hyp3, ordered}}\ (X{\in}1){\vee}(1{\in}X)$
$_{\textit{def}}\ (X{\in}\{0\}){\vee}(1{\in}X)\ _{\textit{equal (hyp1-real)}}\ (X{=}0){\vee}(1{\in}X)\ _{\textit{hyp2-false}}\ 0{\vee}(1{\in}X)\ _{B}\ 1{\in}X$ ■

count limit $_{\textit{indirect}}$: $Z'{\notin}X\ _{\textit{ordered (hyp1+2 subnumber count)}}$
$Z'{\not\subset}X\ _{\textit{false>linear}}\ X{\subseteq}Z'\ _{\textit{monotonic}}\ \bigcup X{\subseteq}\bigcup Z'\ _{\textit{hyp3, countdown (subnumber) +subnumber}}$
$(X{\subseteq}Z){\cdot}(Z{\subseteq}X){\cdot}(Z{\neq}X)\ _{\textit{antisymmtric}}\ (X{=}Z){\cdot}(Z{\neq}X)\ _{B}\ 0$ ■

predecessor:

$_{\textit{subnumber (hyp1)}}\ Z{\in}\mathbb{O}\ _{\textit{countdown limit}}\ (Z{=}\bigcup Z){\vee}(Z{=}(\bigcup Z)')\ _{\textit{+hyp3 B}}\ \underline{Z{=}(\bigcup Z)'}$
$_{\textit{hpy1 countdown}}\ \bigcup Z{\in}\mathbb{O}\ _{\textit{count in}}\ \bigcup Z{\in}(\bigcup Z)'\ _{U}\ \bigcup Z{\in}Z$ ■

successor:

$A{\in}\mathbb{O}\ _{\textit{count in, count}}\ (A{\in}A'){\cdot}(A'{\in}\mathbb{O})\ _{\textit{subnumber}}\ A'{\neq}A\ _{\textit{countdown (hyp)}}\ A'{\neq}\bigcup(A')$ ■

(1) $\mathbb{O}{\subseteq}\mathcal{P}\mathbb{O}$ $_{\textit{free}}$: $X{\in}\mathbb{O}\ _{\textit{real +number set}}\ (X{\in}\mathsf{I}){\cdot}(X{\subseteq}\mathbb{O})\ _{\textit{satified def}}\ X{\in}\mathcal{P}\mathbb{O}$ ■

transfinite induction with $\boldsymbol{K}{:=}\{x|f_x\}$:

$_{\textit{(general)}}$: $_{\textit{premise2>hyp}}\ (f_X{\Rightarrow}f_{X'}){\cdot}((X{=}\bigcup X){\cdot}\forall z{\in}X{:}f_z{\Rightarrow}f_X)\ _{\textit{def satified}}$
$(X{\in}\boldsymbol{K}{\Rightarrow}X'{\in}\boldsymbol{K}){\cdot}((X{=}\bigcup X){\cdot}\forall z{\in}X{:}z{\in}\boldsymbol{K}{\Rightarrow}X{\in}\boldsymbol{K})\ _{\textit{+hyp-distributive, +subnumber-distributive}}$
$\underline{(X{\in}\mathbb{O}{\cdot}\boldsymbol{K}{\Rightarrow}X'{\in}\boldsymbol{K})}{\cdot}((X{=}\bigcup X){\cdot}\forall z{\in}X{\cdot}\mathbb{O}{:}z{\in}\boldsymbol{K}{\Rightarrow}X{\in}\boldsymbol{K})\ _{\textit{subset synonym}}$
$(X{=}\bigcup X){\cdot}(X{\cdot}\mathbb{O}{\subseteq}\boldsymbol{K}){\Rightarrow}X{\in}\boldsymbol{K}\ _{\textit{general (hyp) +premise1 +U-general}}$
$(0{\in}\boldsymbol{K}){\cdot}\forall x{\in}\mathbb{O}{\cdot}\boldsymbol{K}{:}(x'{\in}\boldsymbol{K}){\cdot}\forall x{\in}\mathbb{O}{:}((x{=}\bigcup x){\cdot}(x{\cdot}\mathbb{O}{\subseteq}\boldsymbol{K}){\Rightarrow}x{\in}\boldsymbol{K})$
$_{\textit{induction (reflexive, ordinal, (1))}}\ \mathbb{O}{\subseteq}\boldsymbol{K}\ _{\textit{hyp syllogism}}\ X{\in}\boldsymbol{K}\ _{\textit{def satified}}\ f_X$ ■

9 Explicite recursive definitions

Recursive definitions are usually formulated implicitly using recursion axioms. They can also be explicitly defined as union of sequences with the same recursion rule. If this union is a sequence, the recursion is not transfinite, but stops at an ordinal number. In the transfinite case, this union is defined on all ordinal numbers and is no set; this is the case, if the recursion value exists for any ordinals:

Theorems for terms φ_F, γ_F, Γ, δ:

$\gamma_F := \forall z{\in}\mathrm{Def}F{:}(F(z){=}\varphi_{F\restriction_z})$	*recursion rule*	
$\Gamma := \{f{\in}\mathsf{SEQ}\vert\gamma_f\}$	*recursive funktions*	
$\delta := \bigcup\{f{\in}\mathsf{SEQ}\vert\forall z{\in}\mathrm{Def}f{:}(f(z){=}\varphi_{f\restriction_z})\}$	*recursive definition*	
$\vdash X{\in}\mathrm{Def}\delta \Rightarrow \delta(X){=}\varphi_{\delta\restriction_X}$	*recursion*	[173]
$\mathrm{Def}\delta{=}\{x{\in}\mathbb{O}\vert\exists f{\in}\Gamma{:}(x{\in}\mathrm{Def}f)\}$	*recursive arguments*	[173]
$(X{\in}\mathbb{O}){\cdot}(X{\subseteq}\mathrm{Def}\delta){\Rightarrow}\varphi_{\delta\restriction_X}{\in}\mathsf{I}$	*real values (premise)*	
$\vdash X{\in}\mathbb{O} \Rightarrow (\delta(X){=}\varphi_{\delta\restriction_X})$	*transfinite recursion*	
$\mathrm{Def}\delta{=}\mathbb{O}$	*transfinity*	
$\mathrm{Def}\delta{\notin}\mathsf{I}$, $\mathrm{Def}\delta{\notin}\mathbb{M}$	*totality*	
$X{\in}\mathbb{O} \Rightarrow (X{=}\mathrm{Def}\delta\restriction_X){\cdot}(\delta(X){\in}\mathsf{I})$	*subrecursion*	

Proofs with $G{:=}\delta\restriction_X{\cup}\{(X,\varphi_{\delta\restriction_X}\}$:

(1) $(X{\in}\mathbb{O}){\cdot}\gamma_F{\Rightarrow}\gamma_{F\restriction_X}$ *γ-general*: $Z{\in}\mathrm{Def}(F\restriction_X)$ *restriction* $Z{\in}X{\cdot}\mathrm{Def}F$ *distributive +hyp2-def* $(\underline{Z{\in}X}){\cdot}(Z{\in}\mathrm{Def}F){\cdot}\forall z{\in}\mathrm{Def}F{:}(F(z){=}\varphi_{F\restriction_z})$ *subnumber(hyp1) special* $(Z{=}Z{\cdot}X){\cdot}(F(Z){=}\varphi_{F\restriction_Z})$ *change B* $F(Z){=}\varphi_{F\restriction_{X\cdot Z}}$ *equal value(U), double restriction* $F\restriction_X(Z){=}\varphi_{F\restriction_X\restriction_Z}$ ■

(2) $(X{\in}\mathbb{O}){\cdot}(X{\subseteq}\mathrm{Def}\delta) \Rightarrow (\delta\restriction_X{\in}\mathsf{SEQ}){\cdot}\gamma_{\delta\restriction_X}$: *hyp1 satisfied* $X{\in}\mathbb{M}$ *submap, subfunction(hyp2) extreme* $(\delta\restriction_X{\in}\mathsf{MAP}){\cdot}(X{=}\mathrm{Def}(\delta\restriction_X)){\cdot}(\delta[X]{\subseteq}\mathsf{I})$ *def satisfied* $\delta\restriction_X{:}X{\to}\mathsf{I}$ *def-satisfied (hyp1)* $\delta\restriction_X{\in}\mathsf{SEQ}$; *recursion* $X{\in}\mathrm{Def}\delta{\Rightarrow}(\delta(X){=}\varphi_{\delta\restriction_X})$ *general, def* γ_δ *hyp1 (1)* $\gamma_{\delta\restriction_X}$ ■

(3) $(X{\in}\mathbb{O}){\cdot}(G(X){=}\varphi_{\delta\restriction_X}){\cdot}\forall z{\in}X{:}(G(z){=}\varphi_{\delta\restriction_z}) \Rightarrow \forall z{\in}X'{:}(G(z){=}\varphi_{\delta\restriction_z})$: *general*: $Z{\in}X'$ *smaller* $(Z{\in}X){\vee}(Z{=}X)$ *hyp3-special disjunctive, >hyp2 disjunctive* $(G(Z){=}\varphi_{\delta\restriction_Z}){\vee}(G(Z){=}\varphi_{\delta\restriction_Z})$ *B* $G(Z){=}\varphi_{\delta\restriction_Z}$ ■

(4) $X{\in}\mathbb{O}{\Rightarrow}(X{\subseteq}\mathrm{Def}\delta{\Rightarrow}X{\in}\mathrm{Def}\delta)$: *(2), real values (hyp1+2)* $(\underline{\delta\restriction_X{\in}\mathsf{SEQ}}){\cdot}\underline{\gamma_{\delta\restriction_X}}{\cdot}(\underline{\varphi_{\delta\restriction_X}{\in}\mathsf{I}})$ *subfunction(hyp2)* $X{=}\mathrm{Def}\delta\restriction_X$ *>U2-def, >continuation (U1, U3, def)* $\underline{\forall z{\in}X{:}(\delta\restriction_X(z){=}\varphi_{\delta\restriction_z})}$ ${\cdot}(G{:}X'{\to}\mathsf{I}){\cdot}(\underline{G{\in}\mathsf{SEQ}}){\cdot}\underline{\forall z{\in}X{:}(\delta\restriction_X(z){=}G(z))}{\cdot}(\underline{\varphi_{\delta\restriction_X}{=}G(X)})$ *def-satisfied* $\underline{\mathrm{Def}G{=}X'}$ *>count in (hyp1)* $\underline{X{\in}\mathrm{Def}G}$ *U4+6 quantifier junction, change* $\forall z{\in}X{:}(G(z){=}\varphi_{\delta\restriction_z})$ *+U7, (3)* $\forall z{\in}X'{:}(G(z){=}\varphi_{\delta\restriction_z})$ *U8 def* γ_G *+U5 satisfied* $G{\in}\Gamma$ *+U9 quantify* $\exists f{\in}\Gamma{:}(X{\in}\mathrm{Def}f)$ *+hyp1 satisfied* $X{\in}\{x{\in}\mathbb{O}\vert\exists f{\in}\Gamma{:}(x{\in}\mathrm{Def}f)\}$ *recursive arguments* $X{\in}\mathrm{Def}\delta$ ■

transfinity, totality:

(4) $\forall x{\in}\mathbb{O}{:}(x{\subseteq}\mathrm{Def}\delta{\Rightarrow}x{\in}\mathrm{Def}\delta)$ *transfinite class* $\underline{\mathbb{O}{\subseteq}\mathrm{Def}\delta}$ *recursive arguments def* $\mathrm{Def}\delta{=}\mathbb{O}{\cdot}\{x\vert\exists f{\in}\Gamma{:}(x{\in}\mathrm{Def}f)\}$ *>upper term* $\mathrm{Def}\delta{\subseteq}\mathbb{O}$ *antisymmetric (U)* $\mathbb{O}{=}\mathrm{Def}\delta$ *>totality for* $\mathbb{O}$ $\mathrm{Def}\delta{\notin}\mathsf{I}$, $\mathrm{Def}\delta{\notin}\mathbb{M}$ ■

transfinite recursion, subrecursion:

hyp +number set $(X{\in}\mathbb{O}){\cdot}(X{\subseteq}\mathbb{O})$ *transfinity* $(X{\in}\mathrm{Def}\delta){\cdot}(X{\subseteq}\mathrm{Def}\delta)$ *recursion, subfunction real value (hyp)* $(\delta(X){=}\varphi_{\delta\restriction_X}){\cdot}(X{=}\mathrm{Def}\delta\restriction_X){\cdot}(\varphi_{\delta\restriction_X}{\in}\mathsf{I})$ *change* $\delta(X){\in}\mathsf{I}$ ■

A special case is the transfinite recursion with recursion rules for successor numbers and limit numbers:

Theorems for terms λ_F, σ_F and the *recursive definition* of δ with following φ_F:
$\varphi_F := \sigma_F$ IF $\mathrm{Def}F \neq \bigcup \mathrm{Def}F$ ELSE λ_F
real values $\vdash X \in \mathbb{O} \Rightarrow (\delta(X') = \sigma_{\delta\restriction_{X'}})$ — *successor value*
$(X \in \mathbb{O}) \cdot (X = \bigcup X) \Rightarrow \delta(X) = \lambda_{\delta\restriction_X}$ — *limes value*

Proofs:
successor value:

transfinite recursion + subrecursion (real values, hyp-count) $\overline{(\delta(X') = \varphi_{\delta\restriction_{X'}})} \cdot (X' = \mathrm{Def}\delta\restriction_{X'})$
>successor(hyp) $\mathrm{Def}\delta\restriction_{X'} \neq \bigcup \mathrm{Def}\delta\restriction_{X'}$ *U, conditional definition* $\delta(X') = \sigma_{\delta\restriction_{X'}}$ ■

limes value:

transfinite recursion +subrecursion (real values, hyp1) $\overline{(\delta(X) = \varphi_{\delta\restriction_X})} \cdot (X = \mathrm{Def}\delta\restriction_X)$ *>hyp2*
$\mathrm{Def}\delta\restriction_X = \bigcup \mathrm{Def}\delta\restriction_X$ *U conditional definition* $\delta(X) = \lambda_{\delta\restriction_X}$ ■

10 Class hierarchies

The class hierarchies now discussed generalize Neumann's set hierarchy, which constructs sets by multiple powers of the empty set. One can also start with a class using a **power variant** defined as image of a global swap function that swaps the empty set with this start class.

Definitions:
$S_A := \{(x,y) \mid y = (x \cdot (x \neq A) \vee A \cdot (x = 0))\}$ — *swap function*
$\mathcal{P}_A X := S_A[\mathcal{P}X]$ — *power variant*

Theorems:
$A \in \mathrm{I} \Rightarrow ((X \in \mathrm{I}) \cdot (X \neq A) \cdot (X \neq 0) \Rightarrow (S_A(X) = X))$ — *fix*
$A \in \mathrm{I} \Rightarrow (S_A(0) = A) \cdot (S_A(A) = 0)$ — *swapped*
$A \notin \mathrm{I} \Rightarrow S_A(0) \notin \mathrm{I}$ — *swap gap*
$A \in \mathrm{I} \Rightarrow (X \in \mathrm{I}) \cdot (Y \in \mathrm{I}) \Rightarrow ((S_A(X) = Y) \Rightarrow (X = S_A(Y)))$ — *swap back*
$A \in \mathrm{I} \Rightarrow (S_A \circ S_A = \mathrm{Id})$ — *self-inverse*
$A \in \mathrm{I} \Rightarrow ((0 \in B) \cdot (A \notin B) \Rightarrow (S_A[B] = \{A\} \cup B \setminus \{0\}))$ — *swap*
$A \in B \Rightarrow (0 \in B \Rightarrow (S_A[B] = B))$ — *no swap*
$X \in \mathrm{I} \Rightarrow X \in \{A\} \cup B \setminus \{0\} = (X = A) \vee (X \in B) \cdot (X \neq 0)$ — *swap cases*

Proofs with $\phi_X := X \cdot (X \neq A) \vee A \cdot (X = 0)$, *CASES* $:= (X = A) \vee (X = 0) \vee (X \neq A) \cdot (X \neq 0)$:
(0) CASES: $_B$ ■

(1) $\phi_A=0$: $\phi_A \;_{def}\; A\cdot(A\neq A)\vee A\cdot(A=0) \;_{\textit{reflexive-true, change}}\; A\cdot\neg\mathrm{I}\vee 0\cdot(A=0) \;_{B}\; 0$ ■

(2) $\phi_0=A$: $\phi_0 \;_{def}\; 0\cdot(0\neq A)\vee A\cdot(0=0) \;_{\textit{-reflexive B}}\; A$ ■

(3) $(X\neq A)\cdot(X\neq 0)\Rightarrow(\phi_X=X)$: $\phi_X \;_{def}\; X\cdot(X\neq A)\vee A\cdot(X=0) \;_{\textit{hyp1-true, hyp2-false}}$
$X\cdot\mathrm{I}\vee A\cdot 0 \;_{B}\; X$ ■

(4) $A\in\mathrm{I}\Rightarrow(X\in\mathrm{I}\Rightarrow X\in\mathrm{Def}S_A)$: $_{\textit{(1)(2)(3)disjunctive >CASES}}\; (\phi_X=0)\vee(\phi_X=A)\vee(\phi_X=X)$
$_{\textit{disjunctive>elemental >hyp1 >hyp2, B}}\; \phi_X\in\mathrm{I} \;_{\textit{satisfied-graph2 (hyp2)}}\; X\in\mathrm{Def}S_A$ ■

swapped:
$_{\textit{(4) (hyp, elemental)}}\; (A\in\mathrm{Def}S_A)\cdot(0\in\mathrm{Def}S_A) \;_{\textit{graph1}}\; (S_A(A)=\phi_A)\cdot(S_A(0)=\phi_0)$
$_{\textit{>(1)(2) (hyp)}}\; (S_A(0)=A)\cdot(S_A(A)=0)$ ■

fix: $_{\textit{(4) (hyp 1-4)}}\; X\in\mathrm{Def}S_A \;_{\textit{graph1}}\; S_A(X)=\phi_X \;_{\textit{(3) (hyp3+4)}}\; S_A(X)=X$ ■

swap gap:
$_{\textit{indirect}}$: $S_A(0)\in\mathrm{I} \;_{\textit{argument}}\; 0\in\mathrm{Def}S_A \;_{\textit{graph2-satisfied}}\; \phi_0\in\mathrm{I} \;_{\textit{(2)}}\; A\in\mathrm{I} \;_{\textit{hyp-false}}\; 0$ ■

(5) $(A\in\mathrm{I})\cdot(X\in\mathrm{I})\Rightarrow(S_A(S_A(X))=X)$:
case1 $S_A(S_A(0))=0$: $S_A(S_A(0)) \;_{\textit{swapped (hyp1)}}\; S_A(A) \;_{\textit{swapped (hyp1)}}\; 0$;
case2 $S_A(S_A(A))=A$: $S_A(S_A(A)) \;_{\textit{swapped (hyp1)}}\; S_A(0) \;_{\textit{swapped (hyp1)}}\; A$;
case3 $(X\neq A)\cdot(X\neq 0)\Rightarrow(S_A(S_A(X))=X)$: $S_A(S_A(X))=X \;_{\textit{fix (hyp1+2)}}\; S_A(X)$
$_{\textit{fix (hyp1+2)}}\; X$; $_{\textit{disjunctive: case1-3>CASES, B}}\; S_A(S_A(X))=X$ ■

swap back:
$S_A(X)=Y \;_{\textit{congruent}}\; S_A(S_A(X))=S_A(Y) \;_{\textit{(5)}}\; X=S_A(Y)$ ■

self-inverse:
$S_A\circ S_A \;_{def}\; \{(x,y)|y=S_A(S_A(x))\} \;_{\textit{(5) hyp}}\; \{(x,y)|y=x\} \;_{def}\; \mathrm{Id}$ ■

swap cases:
$X\in\{A\}\cup B\setminus\{0\} \;_{\textit{distributive}}\; (X\in\{A\})\vee(X\in B\setminus\{0\}) \;_{\textit{exept}}$
$(X\in\{A\})\vee(X\in B)\cdot(X\notin\{0\}) \;_{\textit{equal (hyp)}}\; (X=A)\vee(X\in B)\cdot(X\neq 0)$ ■

(6) $(X\in\mathrm{I})\cdot(S_A(X)\in B)=X\in S_A[B]$:
$(X\in\mathrm{I})\cdot(S_A(X)\in B) \;_{\textit{quantified}}\; \exists x{:}((S_A(X)=x)\cdot(x\in B)) \;_{\textit{swap back}}$
$\exists x{:}((X=S_A(x))\cdot(x\in B)) \;_{\textit{def satisfied (hyp1)}}\; X\in S_A[B]$; $_{\textit{reverse: satisfied}}$
$(\underline{X\in\mathrm{I}})\cdot\exists x{:}((X=S_A(x))\cdot(x\in B)) \;_{\textit{swap back}}\; \exists x{:}((S_A(X)=x)\cdot(x\in B)) \;_{\textit{change}}$
$\exists x{:}(S_A(X)\in B) \;_{\textit{not quantified +U}}\; (X\in\mathrm{I})\cdot(S_A(X)\in B)$ ■

swap:
(7) $S_A(X)\in B=(X=A)\vee(X\in B)\cdot(X\neq 0) \;_{\textit{for three cases}}$:
case1 $X=A$: $S_A(A)\in B \;_{\textit{swapped (hyp1)}}\; 0\in B \;_{\textit{hyp2-true}}\; \mathrm{I} \;_{B}\; \mathrm{I}\vee(A\in B)\cdot(A\neq 0) \;_{\textit{reflexive-true}}$
$(A=A)\vee(A\in B)\cdot(A\neq 0)$;
case2 $X=0$: $S_A(0)\in B \;_{\textit{swapped (hyp1) +hyp3}}\; (A\in B)\cdot(A\notin B) \;_{B}\; 0 \;_{B}\; 0\cdot(0=A) \;_{\textit{hyp2-false}}$
$(0\notin B)\cdot(0=A) \;_{\textit{change}}(A\notin B)\cdot(0=A) \;_{\textit{-hyp3}}\; 0=A \;_{B}\; (A=0)\vee\mathrm{I}\cdot 0 \;_{\textit{hyp2-true, reflexive-false}}$

$(0=A)\vee(0\in B)\cdot(0\neq 0)$;

case3 $(X\neq A)\cdot(X\neq 0)$: $S_A(X)\in B$ $_{\textit{fix (case3)}}$ $X\in B$ $_{\textit{+case3/2}}$ $(X\in B)\cdot(X\neq 0)$ $_{B}$ $0\vee(X\in B)\cdot(X\neq 0)$ $_{\textit{case3/1-false}}$ $(X=A)\vee(X\in B)\cdot(X\neq 0)$;

$_{\textit{case1-3>CASES, B}}$ (7) $_{\textit{(6) swap cases}}$ $X\in S_A[B]=X\in\{A\}\cup B\setminus\{0\}$ $_{\textit{free}}$ $S_A[B]=\{A\}\cup B\setminus\{0\}$ ■

no swap:

$_{\textit{free}}$: $X\in S_A[B]$ $_{(6)}$ $S_A(X)\in B$ $_{\textit{disunctive: hyp1 hyp2 swap cases>CASES, B}}$ $X\in B$; $_{\textit{reverse}}$: $\underline{X\in B}$ $_{\textit{disjunctive: swapped/fix>CASES}}$ $(X=S_A(0))\vee(X=S_A(A))\vee(X=S_A(X))$ $_{\textit{quantify (elemental, hyp1, U) disjunctive B}}$ $\exists x{:}((X=S_A(x))\cdot(X\in B))$ $_{\textit{satisfied (hyp1-real hyp2, U)}}$ $X\in S_A[B]$ ■

Of course, only the application of the swap function to the usual power is of interest. The resulting power variant has similar properties and, as a special case, has the usual power:

Theorems:	
$\mathcal{P}_0X=\mathcal{P}X$	*usual power*
$X\subseteq Y\Rightarrow\mathcal{P}_AX\subseteq\mathcal{P}_AY$	*monotonic*
$A\in\mathsf{I}\Rightarrow(X\in\mathbb{B}\Rightarrow(\mathcal{P}_AX\neq F[X])\cdot(\mathcal{P}_AX\neq X))$	*no image*
$X\in\mathbb{B}\Rightarrow\mathcal{P}_AX\in\mathbb{B}$	*real power*
$(A\in\mathsf{I})\cdot(A\notin\mathcal{P}X)\Rightarrow\mathcal{P}_AX=\{A\}\cup\mathcal{P}X\setminus\{0\}$	*swap power*
$A\in\mathcal{P}X\Rightarrow\mathcal{P}_AX=\mathcal{P}X$	*fix power*
$(A\in\mathsf{I})\cdot\forall y\in X{:}(y\subseteq\mathcal{P}_Ay)\Rightarrow\bigcup X\subseteq\mathcal{P}_A\bigcup X$	*accumulation*

Proofs:

usual power:

$\underline{S_0=\mathrm{Id}}$: S_0 $_{\textit{def}}$ $\{(x,y)|y=(x\cdot\neg(x=0)\vee 0\cdot(x=0))\}$ $_{\textit{change}}$ $\{(x,y)|y=(x\cdot\neg(x=0)\vee x\cdot(x=0))\}$ $_{B}$ $\{(x,y)|y=x\}$ $_{\textit{def}}$ Id;

$\mathcal{P}_0X$ $_{\textit{def}}$ $S_0[\mathcal{P}X]$ $_{U}$ $\mathrm{Id}[\mathcal{P}X]$ $_{\textit{identical image}}$ $\mathcal{P}X$ ■

monotonic:

$X\subseteq Y$ $_{\textit{monotonic for }\mathcal{P}}$ $\mathcal{P}X\subseteq\mathcal{P}Y$ $_{\textit{monotonic (for image)}}$ $S_A[\mathcal{P}X]\subseteq S_A[\mathcal{P}Y]$ $_{\textit{def}}$ $\mathcal{P}_AX\subseteq\mathcal{P}_AY$ ■

swap power:

$_{\textit{extreme}}$ $0\subseteq X$ $_{\textit{satisfied (elemental)}}$ $0\in\mathcal{P}X$ $_{\textit{swap (hyp)}}$ $S_A[\mathcal{P}X]=\{A\}\cup\mathcal{P}X\setminus\{0\}$ $_{\textit{def}}$ $\mathcal{P}_AX=\{A\}\cup\mathcal{P}X\setminus\{0\}$ ■

fix power:

$_{\textit{extreme, satisfied (elemental)}}$ $0\in\mathcal{P}X$ $_{\textit{no swap (hyp)}}$ $S_A[\mathcal{P}X]=\mathcal{P}X$ $_{\textit{def}}$ $\mathcal{P}_AX=\mathcal{P}X$ ■

no image:

$_{\textit{indirect}}$: $\mathcal{P}_AX=F[X]$ $_{\textit{congurent, def}}$ $S_A[S_A[\mathcal{P}X]]=S_A[F[X]]$ $_{\textit{image of image}}$

$S_A \circ S_A[\mathcal{P}X]=S_A \circ F[X]$ *self-inverse* $\mathrm{Id}[\mathcal{P}X]=S_A \circ F[X]$ *identical image* $\mathcal{P}X=S_A \circ F[X]$ *no image (for $\mathcal{P}$)-false (hyp1+2)* 0; *special case* $\mathcal{P}_A X \neq \mathrm{Id}[X]$ *identical image* $\mathcal{P}_A X \neq X$ ■

real power:

$X \in \mathbb{B}$ *real power for $\mathcal{P}$* $\mathcal{P}X \in \mathbb{B}$ *real image* $S_A[\mathcal{P}X] \in \mathbb{B}$ *def* $\mathcal{P}_A X \in \mathbb{B}$ ■

accumulation

free: $\underline{Z \in \bigcup X}$ *satisfied* $\exists y:((Z \in y) \cdot (\underline{y \in X}))$ *hyp-special* $\exists y:((Z \in y) \cdot (y \subseteq \mathcal{P}_A y))$ *syllogism* $\underline{\exists y:(\overline{Z \in \mathcal{P}_A y})}$;

case1 $Z=A$: *case2* $A \in \mathcal{P}\bigcup X$: *case1* $Z \in \mathcal{P}\bigcup X$ *fix power (case2)* $Z \in \mathcal{P}_A \bigcup X$;

case $\neg 2$ $A \notin \mathcal{P}\bigcup X$: *case1>hyp-existent* $Z \in \{A\}$ *extensive* $Z \in \{A\} \cup \mathcal{P}\bigcup X \setminus \{0\}$ *swap power (hyp1, case ¬2)* $Z \in \mathcal{P}_A \bigcup X$;

case1-false $(Z=A)=0$ with *case2*: *case3* $A \in \mathcal{P}y$: *fix power* $\mathcal{P}_A y=\mathcal{P}y$ *>U3* $\exists y:(Z \in \mathcal{P}y)$ *upper power(U2)* $\exists y:(Z \in \mathcal{P}\bigcup X)$ *not quantified, fix power (case2)* $Z \in \mathcal{P}_A \bigcup X$;

case $\neg 3$ $A \notin \mathcal{P}y$: *swap power (hyp1) >U3* $\exists y:(Z \in \{A\} \cup \mathcal{P}y \setminus \{0\})$ *distributive, exept* $\exists y:((Z \in \{A\}) \vee (Z \in \mathcal{P}y) \cdot (Z \notin \{0\}))$ *equal (U1-real, elemental)* $\exists y:((Z=A) \vee (Z \in \mathcal{P}y) \cdot (Z \neq 0))$ *case1-false, B* $\exists y:((Z \in \mathcal{P}y) \cdot (Z \neq 0))$ *upper power (U2)* $\exists y:(Z \in \mathcal{P}\bigcup X)$ *not quantified* $Z \in \mathcal{P}\bigcup X$ *fix power (case2)* $Z \in \mathcal{P}_A \bigcup X$ ■

case $\neg 2$: *upper power (contraposition)>U2* $A \notin \mathcal{P}y$ *swap power (hyp1) >U3* $\exists y:(Z \in \{A\} \cup \mathcal{P}y \setminus \{0\})$ *distributive, equal (U1-real)* $\exists y:((Z=A) \vee (Z \in \mathcal{P}y \setminus \{0\}))$ *case1-false, B, exept* $\exists y:((Z \in \mathcal{P}y) \cdot (Z \notin \{0\}))$ *upper power (U2)* $\exists y:((Z \in \mathcal{P}\bigcup X) \cdot (Z \notin \{0\}))$ *not quantified, exept* $Z \in \mathcal{P}\bigcup X \setminus \{0\}$ *extensive* $Z \in \{A\} \cup \mathcal{P}\bigcup X \setminus \{0\}$ *swap power (hyp1,case ¬2)* $Z \in \mathcal{P}_A \bigcup X$ ■

Neumann extended the powers of the empty set through their union into the transfinite using implicitly defined set ranks. An **explicit definition of class ranks** with the power variant achieves the same in a more general form:

Definition:

$V_A := \bigcup\{f \in \mathrm{SEQ} | \forall z \in \mathrm{Def} f:(f(z)=\mathcal{P}_A f(\bigcup z) \text{ IF } z \neq \bigcup z \text{ ELSE } \bigcup f[z])\}$

Theorems:

$V_A(0)=0$	*start rank*
$X \in \mathbb{O} \Rightarrow V_A(X')=\mathcal{P}_A V_A(X)$	*successor rank*
$(X \in \mathbb{O}) \cdot (X=\bigcup X) \Rightarrow V_A(X)=\bigcup V_A[X]$	*limit rank*
$A \in \mathrm{I} \Rightarrow V_A(1)=\{A\}$	*rank 1*
$X \in \mathbb{O} \Rightarrow V_A(X) \in \mathbb{B}$	*real ranks*
$(A \in \mathrm{I}) \cdot (A \notin V_A(X)) \cdot (Y \in \mathcal{P}_A V_A(X)) \Rightarrow$ $(Y=A) \vee (Y \subseteq V_A(X)) \cdot (X \neq 0)$	*subrank-variant*
$(A \in V_A(X)) \cdot (Y \in \mathcal{P}_A V_A(X)) \Rightarrow Y \subseteq V_A(X)$	*subrank*
$X \in \mathbb{O} \Rightarrow Y \in \bigcup V_A[X]=\exists z \in X:(Y \in V_A(z))$	*previous rank*

Proofs with $\sigma_F:=\mathcal{P}_A F(\bigcup \mathrm{Def} F)$, $\lambda_F:=\bigcup \mathrm{Im} F$, $p_F := \mathrm{Def} F\neq\bigcup \mathrm{Def} F$:

(λ) $\lambda_{F\restriction_Z}=\bigcup F[Z]$, $\neg p_{F\restriction_Z}\Rightarrow(\varphi_{F\restriction_Z}=\bigcup F[Z])$:

$_{def}\ \lambda_{F\restriction_Z}=\bigcup \mathrm{Im}(F\restriction_Z)\ _{>restriction}\ \lambda_{F\restriction_Z}=\bigcup F[Z]\ _{conditional\ definition\ (hyp)}\ \varphi_{F\restriction_Z}=\bigcup F[Z]$ ■

(σ) $(Z\in\mathbb{O})\cdot(Z=\mathrm{Def}F\restriction_Z)\cdot p_{F\restriction_Z}\Rightarrow(\sigma_{F\restriction_Z}=\mathcal{P}_A F(\bigcup Z))\cdot(\varphi_{F\restriction_Z}=\mathcal{P}_A F(\bigcup Z))$:

$_{predecessor\ (hyp1,\ hyp3/2)}\ \underline{\bigcup Z\in Z}$; $\varphi_{F\restriction_Z}\ _{conditional\ definition\ (hyp3)}\ \sigma_{F\restriction_Z}\ _{def}$

$\mathcal{P}_A F\restriction_Z(\bigcup \mathrm{Def}(F\restriction_Z))\ _{hyp2}\ \mathcal{P}_A F\restriction_Z(\bigcup Z)\ _{equal\ value\ (hyp1,\ U)}\ \mathcal{P}_A F(\bigcup Z)$ ■

(1) $(F\in\mathrm{SEQ})\cdot(Z\in\mathrm{Def}F)\Rightarrow(\varphi_{F\restriction_Z}=\mathcal{P}_A F(\bigcup Z)$ IF $Z\neq\bigcup Z$ ELSE $\bigcup F[Z])$:

$_{number\ arguments\ (hyp1)}\ \mathrm{Def}F\in\mathbb{O}\ _{subnumber\text{-}subfunction\ (hyp2)}\ \underline{(Z\in\mathbb{O})\cdot(Z=\mathrm{Def}(F\restriction_Z))}$;

$\varphi_{F\restriction_Z}\ _{def}\ \sigma_{F\restriction_Z}\cdot p_{F_Z}\vee\lambda_{F\restriction_Z}\cdot\neg p_{F_Z}\ _{(\lambda)(\sigma)\ disjunctive\ (U)}\ \mathcal{P}_A F(\bigcup Z)\cdot p_{F_Z}\vee\bigcup F[Z]\cdot\neg p_{F_Z}$

$_{def,\ hyp2}\ \mathcal{P}_A F(\bigcup Z)$ IF $Z\neq\bigcup Z$ ELSE $\bigcup F[Z]$ ■

(δ) $\delta=V_A$:

$\delta\ _{def}\ \bigcup\{f\in\mathrm{SEQ}|\forall z\in\mathrm{Def}f:(f(z)=\varphi_{f\restriction_z})\}\ _{(1)}\ V_A$ ■

(φ) $X\in\mathbb{O}\Rightarrow f_X$ with $f_X:=(X\subseteq\mathrm{Def}\delta\Rightarrow\varphi_{\delta\restriction_X}\in\mathbb{B})\ _{transfinite\ induction}$:

base case f_0: $_{subfunction\ (f_0\text{-}hyp)}\ 0=\mathrm{Def}\delta\restriction_0\ _{>zero\ union,\ def\ of\ p}\ \neg p_{\delta_0}\ _{(\lambda)}\ \varphi_{\delta\restriction_0}=\bigcup\delta[0]$

$_{empty\ function}\ \varphi_{\delta\restriction_0}=\bigcup 0\ _{zero\ union}\ \varphi_{\delta\restriction_0}=0\ _{>elemental}\ \varphi_{\delta\restriction_0}\in\mathbb{B}$ ■

successor case $X\in\mathbb{O}\Rightarrow(f_X\Rightarrow f_{X'})$:

$_{f_{X'}\text{-}hyp}\ \underline{X'\subseteq\mathrm{Def}\delta}\ _{subfunction}\ \underline{X'=\mathrm{Def}\delta\restriction_{X'}}\ _{>successor\ (hyp)}\ p_{\delta_X}\ _{(\sigma)\ (hyp\text{-}count,\ U2)}$

$\varphi_{\delta\restriction_{X'}}=\mathcal{P}_A\delta(\bigcup(X'))\ _{countdown\ (hyp)}\ \underline{\varphi_{\delta\restriction_{X'}}=\mathcal{P}_A\delta(X)}$; $_{count\ in\ (hyp)\ +subnumber\ (hyp\text{-}count)}$

$(X\in X')\cdot(X\subseteq X')\ _{syllogism\ (U1),\ transitive\ (U1)}\ (X\in\mathrm{Def}\delta)\cdot(X\subseteq\mathrm{Def}\delta)\ _{recursion,\ hyp2(f_X)}$

$(\delta(X)=\delta\restriction_X)\cdot(\delta\restriction_X\in\mathbb{B})\ _{change}\ \delta(X)\in\mathbb{B}\ _{real\ power\ for\ \mathcal{P}_A}\ \mathcal{P}_A\delta(X)\in\mathbb{B}\ _{U3}\ \varphi_{\delta\restriction_{X'}}\in\mathbb{B}$ ■

limit case $X\in\mathbb{O}\Rightarrow((X=\bigcup X)\cdot\forall z\in X:f_z\Rightarrow f_X)$:

$_{f\text{-}hyp2}\ \underline{X\subseteq\mathrm{Def}\delta}\ _{def\ subfunction}\ X=\mathrm{Def}\delta\restriction_X\ _{hyp2}\ \neg p_{\delta_X}\ _{(\lambda)}\ \underline{\varphi_{\delta\restriction_X}=\bigcup\delta[X]}$;

$_{hyp1\ satisfied}\ X\in\mathbb{M}\ _{subclass}\ X\in\mathbb{B}\ _{real\ image}\ \underline{\delta[X]\in\mathbb{B}}$;

$\underline{\delta[X]\subseteq\mathbb{B}}_{free}$: $Y\in\delta[X]\ _{def\ satisfied}\ \exists z:((\underline{Y=\delta(z)})\cdot(\underline{z\in X}))\ _{syllogism\ (U4),\ subnumber(hyp1)}$

$\exists z:((z\in\mathrm{Def}\delta)\cdot(z\subseteq X))\ _{recursion\ \ transitive\ (U4)}\ \exists z:((\underline{\delta(z)=\varphi_{\delta\restriction_z}})\cdot(z\subseteq\mathrm{Def}\delta))$

$_{(hyp3\text{-}special\ (U9)}\ \exists z:(\varphi_{\delta\restriction_z}\in\mathbb{B})\ _{U10,\ U8,\ not\ quantified}\ Y\in\mathbb{B}$;

$_{real\ union\ (U6+7)}\ \bigcup\delta[X]\in\mathbb{B}\ _{U5}\ \varphi_{\delta\restriction_X}\in\mathbb{B}$ ■

real ranks and *real values*:

$X\in\mathbb{O}\ _{(\varphi)}\ X\subseteq\mathrm{Def}\delta\Rightarrow\varphi_{\delta\restriction_X}\in\mathbb{B}\ _{>hyp2}\ \underline{\varphi_{\delta\restriction_X}\in\mathbb{B}}\ _{real}\ \varphi_{\delta\restriction_X}\in\mathrm{I}$ ■

$_{U\ transfinite\ recursion}\ \delta(X)\in\mathbb{B}\ _{(\delta)}\ V_A(X)\in\mathbb{B}$ ■

start rank:

$_{ordinal}\ 0\in\mathbb{O}\ _{transfinity\ (real\ values)}\ 0\in\mathrm{Def}\delta\ _{limes\ value\ (zero\ union)}\ \delta(0)=\lambda_{\delta\restriction_0}\ _{(\delta)(\lambda)}$

$V_A(0)=\bigcup\delta[0]\ _{empty\ function}\ V_A(0)=0$ ■

successor rank:

$_{\textit{hyp-count, subrecursion}}\ \underline{X'=\mathrm{Def}(\delta\restriction_{X'})}\ _{\textit{>successor (hyp)}}\ \underline{p_\delta}$;
$V_A(X')\ _{(\delta)}\ \delta(X')\ _{\textit{successor value (real values, hyp)}}\ \sigma_{\delta\restriction_{X'}}\ _{(\sigma)\ \textit{(hyp-count, U1, U2)}}$
$\mathcal{P}_A\delta(\bigcup(X'))\ _{\textit{countdown (hyp), }(\delta)}\ \mathcal{P}_A V_A(X)$ ■

limit rank:

$V_A(X)\ _{(\delta)}\ \delta(X)\ _{\textit{limes value (real values, hyp)}}\ \lambda_{\delta\restriction_X}\ _{(\lambda)}\ \bigcup\delta[X]\ _{(\delta)}\ \bigcup V_A[X]$ ■

rank 1:

$V_A(1)\ _{\textit{def}}\ V_A(0')\ _{\textit{successor rank}}\ \mathcal{P}_A V_A(0)\ _{\textit{start rank}}\ \mathcal{P}_A 0\ _{\textit{def}}\ S_A[\mathcal{P}0]\ _{\textit{zero power}}\ S_A[\{0\}]$
$_{\textit{single image}}\ \{S_A(0)\}\ _{\textit{swapped (hyp)}}\ \{A\}$ ■

subrank-variant:

$Y\in\mathcal{P}_A V_A(X)\ _{\textit{swap (hyp1)}}\ Y\in\{A\}\vee\mathcal{P}V_A(X)\setminus\{0\}\ _{\textit{swap cases}}$
$(Y=A)\vee(Y\in\mathcal{P}V_A(X))\cdot(Y\neq 0)\ _{\textit{satisfied}}\ (Y=A)\vee(Y\subseteq V_A(X))\cdot(Y\neq 0)$ ■

subrank:

$Y\in\mathcal{P}_A V_A(X)\ _{\textit{no swap}}\ Y\in\mathcal{P}V_A(X)\ _{\textit{satisfied}}\ Y\subseteq V_A(X)$ ■

previous rank:

$Y\in\bigcup V_A[X]\ _{\textit{value element}}\ \exists\underline{z\in X}:((Y\in V_A(z))\cdot(V_A(z)\in\mathrm{I}))\ _{\textit{-real rank (hyp U subnumber)}}$
$\exists z\in X:(Y\in V_A(z))$ ■

Each class can be chosen as a start class. However, in the case of non-existent classes such as the Russell class, the class of all ordinals or the class of all sets, only empty ranks and empty hierarchies result. Effective **cumulative hierarchies**, whose ranks form an ascending chain of inclusion of mappable classes, only result from the union of all ranks if there is an existing start class:

Definition:
THE HIERARCHY ABOVE $A := \mathbb{V}_A := \bigcup V_A[\mathbb{O}]$

Theorems:

$A\notin\mathrm{I}\Rightarrow\mathbb{V}_A=0,\ \mathbb{V}_\dagger=0,\ \mathbb{V}_\mathbb{M}=0,\ \mathbb{V}_\mathbb{O}=0$	*empty hiearachies*
$\mathbb{V}_A\subseteq\mathbb{B}\cup\{A\},\quad A\in\mathbb{B}\Rightarrow\mathbb{V}_A\subseteq\mathbb{B}$	*mappable*
$A\in\mathrm{I}\Rightarrow\forall x\in\mathbb{O}:\forall z\in x:(V_A(z)\subset V_A(x))$	*cumulative hierachy*

Proofs:

(0): $A\notin\mathrm{I}\Rightarrow\forall x\in\mathbb{O}:(V_A(x)=0)$ with $f_X:=(V_A(X)=0)\ _{\textit{transfinit induction}}$:

base case f_0: $_{\textit{start rank}}\ V_A(0)=0$ ■

successor case $X\in\mathbb{O}\Rightarrow(f_X\Rightarrow f_{X'})$: $V_A(X')\ _{\textit{successor rank}}\ \mathcal{P}_A V_A(X)\ _{\textit{hyp2-def}}\ \mathcal{P}_A 0\ _{\textit{def}}$
$S_A[\mathcal{P}0]\ _{\textit{zero power}}\ S_A[\{0\}]\ _{\textit{single image}}\ \{S_A(0)\}\ _{\textit{swap gap, not existent}}\ 0$ ■

limit case $X\in\mathbb{O}\Rightarrow((X=\bigcup X)\cdot\forall z\in X:f_z\Rightarrow f_X)$:

case $X=0$: *base case* f_0; *case* $\underline{X\neq 0}$: $V_A(X)$ *limit rank (hyp1+2)* $\bigcup V_A[X]$ *empty function (hyp3, case)* $\bigcup\{0\}$ *unit union (elemental)* 0 ■

empty hierarchy:
$\mathbb{V}_A$ *def* $\bigcup\{y|\exists x:((y=V_A(x))\cdot(x\in\mathbb{O}))\}$ *(0)(hyp)* $\bigcup\{y|\exists x:((y=0)\cdot(x\in\mathbb{O}))\}$ *not quantified* $\bigcup\{y|(y=0)\cdot\exists x:(x\in\mathbb{O})\}$ *not-empty* $\bigcup\{y|(y=0)\cdot(\mathbb{O}\neq 0)\}$ *-ordinal not-empty* $\bigcup\{y|y=0\}$ *def* $\bigcup\{0\}$ *unit union (elemental)* 0 ■

(1) $X\in\mathbb{O}\Rightarrow(Y\in V_A(X))\cdot(Y\neq A)\Rightarrow Y\in\mathbb{B}$:
transfinite induction with $f_X:=(Y\in V_A(X))\cdot(Y\neq A)\Rightarrow Y\in\mathbb{B}$:
base case f_0: *f_0-hyp1* $Y\in V_A(0)$ *start rank* $Y\in 0$ *empty-false* 0 *B(ex falso quodlibet)* $Y\in\mathbb{B}$ ■
successor case $X\in\mathbb{O}\Rightarrow(f_X\Rightarrow f_{X'})$: *$f_{X'}$-hyp* $Y\in V_A(X')$ *successor rank* $\underline{Y\in\mathcal{P}_A V_A(X)}$ *U1 case1 $A\notin V_A(X)$ subrank-variant* $(Y=A)\vee(Y\subseteq V_A(X))\cdot(X\neq 0)$ *f_X-hyp2-false B* $Y\subseteq V_A(X)$; *U1 case2 $A\in V_A(X)$ subrank* $Y\subseteq V_A(X)$; *case1+2 +real rank, real part* $Y\in\mathbb{B}$ ■
limit case $X\in\mathbb{O}\Rightarrow((X=\bigcup X)\cdot\forall z\in X:f_z\Rightarrow f_X)$: *f-hyp1* $Y\in V_A(X)$ *limit rank* $Y\in\bigcup V_A[X]$ *previous rank* $\exists z:((\underline{Y\in V_A(z)})\cdot(z\in X))$ *hyp3-special f_z >f-hyp2+U2* $Y\in\mathbb{B}$ ■

mappable:
free: *case1* $X=A$: $X\in\mathbb{V}_A$ *real-existent* $X\in\{X\}$ *case1* $X\in\{A\}$ *extensive* $X\in X\in\mathbb{B}\cup\{A\}$;
case2 $X\neq A$: $X\in\mathbb{V}_A$ *def, previous rank* $\exists z:((\underline{X\in V_A(z)})\cdot(z\in\mathbb{O}))$ *(1) >U1, case2* $\exists z:(X\in\mathbb{B})$ *not quantified* $X\in\mathbb{B}$ *extensive* $X\in\mathbb{B}\cup\{A\}$ ■
$A\in\mathbb{B}$ *individual def* $\{A\}=\{A\}\cdot\mathbb{B}$ *congruent* $\mathbb{B}\cup\{A\}=\mathbb{B}\cup\{A\}\cdot\mathbb{B}$ *B* $\mathbb{B}\cup\{A\}=\mathbb{B}$ *>mappable1* $\mathbb{V}_A\subseteq\mathbb{B}$ ■

(2) $(A\in\mathrm{I})\cdot(X\in\mathbb{O})\Rightarrow V_A(X)\subseteq\mathcal{P}_A V_A(X)$ with $f_X:=V_A(X)\subseteq\mathcal{P}_A V_A(X)$ *transfinit induction*:
base case f_0: *extreme* $0\subseteq\mathcal{P}_A V_A(0)$ *start rank* $V_A(0)\subseteq\mathcal{P}_A V_A(0)$ ■
successor case $X\in\mathbb{O}\Rightarrow(f_X\Rightarrow f_{X'})$: $V_A(X)\subseteq\mathcal{P}_A V_A(X)$ *monotonic* $\mathcal{P}_A V_A(X)\subseteq\mathcal{P}_A\mathcal{P}_A V_A(X)$ *sucessor rank* $V_A(X')\subseteq\mathcal{P}_A V_A(X')$ ■
limit case $X\in\mathbb{O}\Rightarrow((X=\bigcup X)\cdot\forall z\in X:f_z\Rightarrow f_X)$: *hyp3* $\forall z\in X:V_A(z)\subseteq\mathcal{P}_A V_A(z)$ *accumulation (hyp)* $\bigcup V_A[X]\subseteq\mathcal{P}_A\bigcup V_A[X]$ *limit rank* $V_A(X)\subseteq\mathcal{P}_A V_A(X)$ ■

(3) $(Z\in X)\cdot(X\in\mathbb{O})\Rightarrow V_A(Z)\subseteq\bigcup V_A[X]$:
free: $Y\in V_A(Z)$ *quantifiy (hyp1)* $\exists z\in X:(Y\in V_A(z))$ *previous rank (hyp2)* $Y\in\bigcup V_A[X]$ ■

cumulative hierachy: with $f_X:=\forall z\in X:(V_A(z)\subset V_A(X))$ *transfinit induction*:
base case f_0 *free*: $Z\in 0$ *+not empty* $(Z\in 0)\cdot(Z\notin 0)$ *B(ex falso quodlibet)* $V_A(Z)\subset V_A(0)$ ■
successor case $X\in\mathbb{O}\Rightarrow(f_X\Rightarrow f_{X'})$: *(2) (hyp) +no image (real ranks)* $(V_A(X)\subseteq\mathcal{P}_A V_A(X))\cdot(V_A(X)\neq\mathcal{P}_A V_A(X))$ *def, successor rank (hyp1)* $\underline{V_A(X)\subset V_A(X')}$;
$f_{X'}$ *general*: $Z\in X'$ *smaller (hyp1)* $(Z\in X)\vee(Z=X)$ *f_X-special (=hyp2) disjunctive* $(V_A(Z)\subset V_A(X))\vee(Z=X)$ *disjunctive: U-transitive, >U, B* $V_A(Z)\subset V_A(X')$ ■

limit case detailed $X{\in}\mathbb{O}{\Rightarrow}((X{=}\bigcup X){\cdot}\forall y{\in}X{:}\forall z{\in}y{:}(V_A(z){\subset}V_A(y)){\Rightarrow}f_X)$:
$f_{X\ general}$: $Z{\in}X$ $_{\text{cout in (subnumber), count limit (hyp1+2)}}$ $(Z{\in}Z'){\cdot}(Z'{\in}X)$ $_{\text{hyp3-special(hyp2) +(3)}}$
$(V_A(Z){\subset}V_A(Z')){\cdot}(V_A(Z'){\subseteq}\bigcup V_A[X])$ $_{\text{transitive}}$ $V_A(Z){\subset}\bigcup V_A[X]$ $_{\text{limit rank (hyp1+2)}}$
$V_A(Z){\subset}V_A(X)$ ∎

Neumann's set hierarchy is apparently the case V_0 with the *usual power* $\mathcal{P}_0X{=}\mathcal{P}X$ without swap. With simplified notation one gets the following corollary:

Definitions:
$V\ := V_0$
NEUMANN'S HIERARCHY $:= \mathbb{V}_0$

Theorems:

$V(0){=}0$	*start set rank*
$X{\in}\mathbb{O} \Rightarrow V(X'){=}\mathcal{P}V(X)$	*successor set rank*
$(X{\in}\mathbb{O}){\cdot}(X{=}\bigcup X) \Rightarrow V(X){=}\bigcup V[X]$	*limit set rank*
$V(1){=}\{0\}$	*set rank 1*
$V(2){=}\{0, \{0\}\}$	*set rank 2*
$V(3){=}\{0, \{0\}, \{\{0\}\}, \{0,\{0\}\}\}$	*set rank 3*
$\mathbb{V}_0{\subseteq}\mathbb{M}$	*set hierarchy*

The hierarchy schema also applies to urelements; with a simple transfinite induction you get a larger hierarchy (with the usual power) that includes $\mathbb{V}_0$ as a proper subclass:

Theorems:

$(A{=}\{A\}){\cdot}(X{\in}\mathbb{O}){\cdot}(1{\in}X) \Rightarrow V_A(X'){=}\mathcal{P}V_A(X)$	*usual power ranks*
$(A{=}\{A\}) \Rightarrow V_A(1){=}\{A\}$	*urelement rank 1*
$(A{=}\{A\}) \Rightarrow V_A(2){=}\{0, A\}$	*urelement rank 2*
$(A{=}\{A\}) \Rightarrow V_A(3){=}\{0, A, \{0\}, \{0, A\}\}$	*urelement rank 3*

Here, however, hierarchies are interested in which the empty set is swapped at each step. Examples are $\mathbb{V}_1$, $\mathbb{V}_2$. Such **regular hierarchies** result, when the start class is no subclass of a rank:

Theorem:
$(A{\in}\mathbb{I}){\cdot}\forall x{\in}\mathbb{O}{:}(A{\not\subseteq}V_A(x)) \Rightarrow$

$X{\in}\mathbb{O} \Rightarrow V_A(X'){=}\{A\}{\cup}\mathcal{P}V_A(X){\setminus}\{0\}$	*regular rank*
$V_A(1){=}\{A\}$	*rank 1*
$V_A(2){=}\{A,\{A\}\}$	*rank 2*
$V_A(3){=}\{A,\{A\},\{\{A\}\},\{A,\{A\}\}\}$	*rank 3*

Proofs:

regular rank:

$_{\textit{hyp2-special (hyp3) def}}\ \neg(A\subseteq V_A(X))\ _{\textit{satisfied (hyp1), def}}\ A\notin\mathcal{P}V_A(X)\ _{\textit{swap power (hyp1)}}$
$\mathcal{P}_A V_A(X)=\{A\}\cup\mathcal{P}V_A(X)\setminus\{0\}\ _{\textit{sucessor rank (hyp3)}}\ V_A(X')=\{A\}\cup\mathcal{P}V_A(X)\setminus\{0\}$ ■

(1) $0\notin X \Rightarrow X=X\setminus\{0\}$:
$0\notin X\ _{\textit{def}}\ \{0\}\cdot X=0\ _{\textit{congruent}}\ X\setminus\{0\}\cup\{0\}\cdot X=X\setminus\{0\}\cup 0\ _{B}\ X=X\setminus\{0\}$ ■

(2) $A\in\mathsf{I}\Rightarrow 0\notin\{\{A\}\}$: $_{\textit{indirect}}\ 0\in\{\{A\}\}\ _{\textit{equal}}\ 0=\{A\}\ _{\textit{hyp-existent-false}}\ 0$ ■

(3) $A\in\mathsf{I}\Rightarrow 0\notin\{\{A\},\{\{A\}\},\{A,\{A\}\}\}$: $_{\textit{indirect}}\ 0\in\{\{A\},\{\{A\}\},\{A,\{A\}\}\}\ _{\textit{ambiguous}}$
$(0=\{A\})\vee(0=\{\{A\}\})\vee(0=\{A,\{A\}\})\ _{\textit{disjunctive: hyp-existent-false, elemental-false, >listet(hyp), B}}$
$A\in 0\ _{\textit{empty-false, B}}\ 0$ ■

rank 2:
$V_A(2)\ _{\textit{def}}\ V_A(1')\ _{\textit{regular rank (hyp1 hyp2-special(ordinal))}}\ \{A\}\cup\mathcal{P}V_A(1)\setminus\{0\}\ _{\textit{rank 1}}$
$\{A\}\cup\mathcal{P}\{A\}\setminus\{0\}\ _{\textit{unit power}}\ \{A\}\cup\{0,\{A\}\}\setminus\{0\}\ _{\textit{def B}}\ \{A\}\cup\{\{A\}\}\setminus\{0\}\ _{\textit{(2)(1)}}$
$\{A\}\cup\{\{A\}\}\ _{\textit{def}}\ \{A,\{A\}\}$ ■

rank 3:
$V_A(3)\ _{\textit{def}}\ V_A(2')\ _{\textit{regular rank (hyp1 hyp2-special(ordinal)}}\ \{A\}\cup\mathcal{P}V_A(2)\setminus\{0\}\ _{\textit{rank 2}}$
$\{A\}\cup\mathcal{P}\{A,\{A\}\}\setminus\{0\}\ _{\textit{pair power}}\ \{A\}\cup\{0,\{A\},\{\{A\}\},\{A,\{A\}\}\}\setminus\{0\}\ _{\textit{B def}}$
$\{A\}\cup\{\{A\},\{\{A\}\},\{A,\{A\}\}\}\setminus\{0\}\ _{\textit{(3)(1) def}}\ \{A,\{A\},\{\{A\}\},\{A,\{A\}\}\}$ ■

Of course this hierarchy scheme also works when transcending set theory. In Leibniz's binary logic not only the empty set 0 exists, but also its complement I, which produces in usual set theory Cantor's antinomy [97], because there is a set universe assumed: only the assumption $\mathbb{M}=\mathsf{I}$ generates the antinomy. With Ockham's razor we move this arbitrary assumption, because all essential results of set theory are provable in the extensional universe even without it. There one can consider the universe as existent without risk and thus gets Leibniz's binary logic, for which there are some equivalent axioms:

Definition:
TRANSCENDENT := $\mathbb{T}$:= NOT IMAGE

Theorems in $\mathbb{E}$:

Cantor or ZF set theory $\vdash \mathbb{M}=\mathsf{I} \vdash \mathsf{I}\notin\mathsf{I}$		*set universe*	[96f]
$\mathbb{L}$ ⊣⊢ I IS TRUE	⊣⊢ I IS EXISTENT	*truth exists*	
$\mathbb{L}$ ⊣⊢ I IS TRUE AND 0 IS FALSE		*binary logic*	
$\mathbb{L}$ ⊣⊢ $\mathbb{L}=\mathsf{I}$	⊣⊢ $\mathbb{L}=\mathbb{E}$	*Leibniz universe*	
$\mathbb{L}$ ⊣⊢ $\mathsf{I}\in\mathbb{L}$	⊣⊢ I IS TRANSCENDENT	*transcendence*	

Proofs:

truth exists:

$\mathbb{L}\ _{\textit{def binary synonym B, falsifiability def}}\ (\mathsf{I}\in\{\mathsf{I}\})\cdot\mathbb{E}\ _{\textit{-premise }\mathbb{E}\textit{, def}}\ \mathsf{I}\ \text{IS TRUE}\ \blacksquare$

binary logic:

$\mathbb{L}\ _{\textit{def binary synonym B}}\ (\mathsf{I}\in\{\mathsf{I}\})\cdot(0\in\{0\})\cdot\mathbb{E}\ _{\textit{-premise }\mathbb{E}\textit{, def}}\ \mathsf{I}\ \text{IS TRUE AND}\ 0\ \text{IS FALSE}\ \blacksquare$

Leibniz universe:

$\mathbb{L}\ _{\textit{judgement}}\ \mathbb{L}=\mathsf{I}\ _{\textit{+premise}}\ \mathbb{L}=\mathsf{I}\cdot\mathbb{E}\ _{B}\ \mathbb{L}=\mathbb{E}\ \blacksquare$

(2) $(\mathsf{I}=F[X])\Rightarrow(\mathsf{Id}\restriction_{\mathcal{P}X}\circ F[X]=\mathcal{P}X)$:

$\mathsf{Id}\restriction_{\mathcal{P}X}\circ F[X]\ _{\textit{image of image}}\ \mathsf{Id}\restriction_{\mathcal{P}X}[F[X]]\ _{\textit{hyp}}\ \mathsf{Id}\restriction_{\mathcal{P}X}[\mathsf{I}]\ _{\textit{def}}\ \mathsf{Im}(\mathsf{Id}\restriction_{\mathcal{P}X})\ _{\textit{restriction}}\ \mathsf{Id}[\mathcal{P}X]\ _{\textit{identical image}}\ \mathcal{P}X\ \blacksquare$

(3) $\mathsf{I}\notin\mathbb{B}\ _{\textit{indirect}}$: $\mathsf{I}\in\mathbb{B}\ _{\textit{satisfied}}\ \exists f{:}((\mathsf{I}=f(x))\cdot(x\in\mathbb{M}))\ _{(2)}\ \exists f{:}((\mathsf{Id}\restriction_{\mathcal{P}x}\circ f[x]=\mathcal{P}X)\cdot(x\in\mathbb{M}))\ _{\textit{no image-false}}\ \exists f{:}(0\cdot(x\in\mathbb{M}))\ _{\textit{not quantified B}}\ 0\ \blacksquare$

transcendence:

$\mathbb{L}\ _{\textit{truth exists, def}}\ \mathsf{I}\in\mathsf{I}\ _{\textit{Leibniz univers}}\ \mathsf{I}\in\mathbb{L}\ _{\textit{real +(3)}}\ (\mathsf{I}\in\mathsf{I})\cdot(\mathsf{I}\notin\mathbb{B})\ _{\textit{exept}}\ \mathsf{I}\in\neg\mathbb{B}\ _{\textit{def}}\ \mathsf{I}\in\mathbb{T}\ _{\textit{real}}\ \mathsf{I}\in\mathsf{I}\ _{\textit{truth exists}}\ \mathbb{L}\ \blacksquare$

The existence of the truth affects the theologic of the *Monadology* [1], where Leibniz gives a valid proof of God with a precise definition equivalent to $\text{GOD}=\{\mathsf{I}\}$, analyzed in a section on an universal philosophical language in [5] [240] and already in [3]. But in his binary logic, the existence of truth has a far greater effect: it creates the **truth hierarchy**. This is a immanent world created with swapped truth values. It is as powerful as the set world because it is a bijective projection of Neumann's hierarchy into a class of mappable nonsets. This follows as a corollary from a general theorem for hierarchies generated by a transcendent class:

Theorems:

$A\in\mathbb{T}\Rightarrow(\mathbb{M}\subset\mathbb{B})\cdot(\mathbb{B}\subset\mathsf{I})$	*higher universe*
$A\in\mathbb{T}\Rightarrow V_A(X')=\{A\}\cup\mathcal{P}V_A(X)\setminus\{0\}$	*regular alien ranks*
$A\in\mathbb{T}\Rightarrow(\mathbb{V}_A\subseteq\neg\mathbb{M})\cdot(\mathbb{V}_A\subseteq\neg\mathbb{V}_0)$	*alien worlds*

Theorems in $\mathbb{L}$:

I IS TRANSCENDENT	*transcendence*
$X\in\mathbb{O}\Rightarrow V_{\mathsf{I}}(X')=\{\mathsf{I}\}\cup\mathcal{P}V_{\mathsf{I}}(X)\setminus\{0\}$	*regular truth rank*
$V_{\mathsf{I}}(1)=\{\mathsf{I}\}$	*truth rank 1*
$V_{\mathsf{I}}(2)=\{\mathsf{I},\{\mathsf{I}\}\}$	*truth rank 2*
$V_{\mathsf{I}}(3)=\{\mathsf{I},\{\mathsf{I}\},\{\{\mathsf{I}\}\},\{\mathsf{I},\{\mathsf{I}\}\}\}$	*truth rank 3*
$\mathbb{V}_{\mathsf{I}}\subseteq\mathbb{B}\cup\{\mathsf{I}\}$	*immanent thruth hierarchy*
$(\mathbb{V}_{\mathsf{I}}\subseteq\neg\mathbb{M})\cdot(\mathbb{V}_{\mathsf{I}}\subseteq\neg\mathbb{V}_0)$	*supernatural world*

Proofs:

(1) $A\in\mathbb{T}\Rightarrow A\notin\mathbb{M}$:

$_{\textit{indirect}}\ A\in\mathbb{M}\ _{\textit{subclass}}\ A\in\mathbb{B}\ _{\textit{hyp-def-exept-false}}\ 0$ ■

higher universe:

$\underline{\mathbb{B}\neq\mathbb{I}}\ _{\textit{indirect}}$: $\mathbb{B}=\mathbb{I}\ _{\textit{>hyp}}\ A\notin\mathbb{I}\ _{\textit{hyp-real-false}}\ 0$; $_{\textit{extreme +U1}}\ (\mathbb{B}\subseteq\mathbb{I})\cdot(\mathbb{B}\neq\mathbb{I})\ _{\textit{def}}\ \mathbb{B}\subset\mathbb{I}$ ■

$\underline{\mathbb{M}\neq\mathbb{B}}\ _{\textit{indirect}}$: $\mathbb{M}=\mathbb{B}\ _{\textit{>elemental}}\ \{A\}\in\mathbb{M}\ _{\textit{sets of sets +hyp-existent}}\ (A\in\{A\})\cdot(\{A\}\subseteq\mathbb{M})$
$_{\textit{syllogism}}\ A\in\mathbb{M}\ _{\textit{(1)-false}}\ 0$; $_{\textit{subclass +U3}}\ (\mathbb{M}\subseteq\mathbb{B})\cdot(\mathbb{M}\neq\mathbb{B})\ _{\textit{def}}\ \mathbb{M}\subset\mathbb{B}$ ■

(2) $A\notin\mathbb{B}\Rightarrow\forall x\in\mathbb{O}:(A\nsubseteq V_A(x))$.

$_{\textit{general, indirect}}\ A\subseteq V_A(X)\ _{\textit{real ranks}}\ (A\subseteq V_A(X))\cdot(V_A(X)\in\mathbb{B})\ _{\textit{real part}}\ A\in\mathbb{B}\ _{\textit{hyp1-false}}\ 0$ ■

regular alien ranks:

$A\in\mathbb{T}\ _{\textit{def}}\ A\in\neg\mathbb{B}\ _{\textit{exept}}\ (A\in\mathbb{I})\cdot(A\notin\mathbb{B})\ _{(2)}\ (A\in\mathbb{I})\cdot\forall x\in\mathbb{O}:(A\nsubseteq V_A(x))\ _{\textit{regular rank}}$
$V_A(X')=\{A\}\cup\mathcal{P}V_A(X)\setminus\{0\}$ ■

(3) $(R\subseteq\neg\mathbb{M})\cdot(R\neq 0)\Rightarrow R\notin\mathbb{M}$:

$_{\textit{indirect}}$: $R\in\mathbb{M}\ _{\textit{sets of sets}}\ R\subseteq\mathbb{M}\ _{\textit{def +hyp1-def}}\ (R=R\cdot\mathbb{M})\cdot(R=R\cdot\neg\mathbb{M})\ _{\textit{change}}$
$R=R\cdot\neg\mathbb{M}\cdot\mathbb{M}\ _{B}\ R=0\ _{\textit{hyp2-false}}\ 0$ ■

(4) $A\in\mathbb{T}\ \Rightarrow\ X\in\mathbb{O}\Rightarrow(V_A(X)\subseteq\neg\mathbb{M})$ with $f_X:=V_A(X)\subseteq\neg\mathbb{M}\ _{\textit{transfinite induction}}$:

base case f_0: $_{\textit{extrem}}\ 0\subseteq\neg\mathbb{M}\ _{\textit{start rank}}\ V_A(0)\subseteq\neg\mathbb{M}$ ■

successor case $X\in\mathbb{O}\Rightarrow(f_X\Rightarrow f_{X'})$: $f_{X'}\ _{\textit{free}}$: $\underline{Y\in V_A(X')}\ _{\textit{regular alien rank,}}$
$_{\textit{subrank-variant}}\ (Y=A)\vee(Y\subseteq V_A(X))\cdot(X\neq 0)\ _{\textit{disjunctive: sucessor case-hyp2 transitive}}$
$(Y=A)\vee(Y\subseteq\neg\mathbb{M})\cdot(X\neq 0)\ _{\textit{disjunctive: (1)(hyp) (3), B}}\ Y\notin\mathbb{M}\ _{\textit{exept (U-real)}}\ Y\in\neg\mathbb{M}$ ■

limit case $X\in\mathbb{O}\Rightarrow((X=\bigcup X)\cdot\forall z\in X:f_z\Rightarrow f_X)$: $_{f_X\,\textit{free}}$: $Y\in V_A(X)\ _{\textit{limit rank}}$
$Y\in\bigcup V_A[X]\ _{\textit{previous rank}}\ \exists z\in X:(Y\in V_A(z))\ _{\textit{+hyp3-special } f_z \textit{ syllogism}}\ Y\in\neg\mathbb{M}$ ■

alien worlds:

$_{\textit{free}}$: $X\in\mathbb{V}_A\ _{\textit{def}}\ X\in\bigcup V_A[\mathbb{O}]\ _{\textit{previous rank (hyp)}}\ \exists z\in\mathbb{O}:(X\in V_A(z))\ _{\textit{+(4) syllogism, not quantified}}$
$X\in\neg\mathbb{M}$ ■

$_{\textit{alien worlds1}}\ \mathbb{V}_A\subseteq\neg\mathbb{M}\ _{\textit{+set hierarchy contraposition(B)}}\ (\mathbb{V}_A\subseteq\neg\mathbb{M})\cdot(\neg\mathbb{M}\subseteq\neg\mathbb{V}_0)\ _{\textit{transitive}}$
$\mathbb{V}_A\subseteq\neg\mathbb{V}_0$ ■

The truth hierarchy is also part of Peano's complementary logic $\mathbb{K}$, in which $\mathbb{L}$ has already been proved. In his logic complements of all sets and all real classes exist. Thus there are immensely more potential images and isomorphic hierarchies. From the point of view of ususl set theory, which only encompasses the lowest hierarchy, this is a logical **multiverse embedded in teh transcendent universe**.

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