Irrationality of the square root of 2.

Proofs in a universal logic calculus

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1 Introduction

One of the oldest famous proofs, recorded in the last proposition of Book X of Euclid's *Elements*, demonstrates that the diagonal and side of a square have an irrational ratio. Today we express this by saying that $\sqrt{2}$ ist irrational. To translate this proof into a modern logic calculus is a popular quick test of the visual quality of this calculus; see, for example, articles by Wiedijk [5, 6] or Cramer et al. [1].

In this note, this proof is translated into a the universal logic calculus described in my book *Universallogik* (Neumaier [3]). For a quick overview of this logic and the underlying idea you may look at the table of contents on the above website.

This new calculus is characterized by four special features:

- It is based on the usual mathematical notation.
- It integrates the historically grown logical language including set theory.
- It is nevertheless axiomatically extremely simple.
- Everything can be precisely formulated verbally (at present in German only) using a logical grammar defined in the *Verbale Logik* (Neumaier [4]).

The simplicity and power of this universal logic are based on a new approach to logic that takes up Aristotelian term logic and perfects it with ideas from Leibniz and Peano. This logic uses four base symbols $=$, $\cdot$, $\neg$, I in four base terms which can be verbalized as follows:

Verbalizations:

A IS IDENTICAL TO $B := (A{=}B)$ — *identity*
A AND $B := A{\wedge}B := A{\cap}B := A{\cdot}B$ — *conjunction*
NOT $A := \neg A$ — *negation*
EXISTENT $:= \mathsf{I}$ — *maximum term*

These base terms satisfy the rules of a Boolean lattice, in my universal logic in the compact version according to Leibniz [3, p. 49]. They produce the logical language of the Aristotelian tradition with free variables for any terms, here always written with capital letters. It is typical for a term logic that propositions count as terms and also satisfy the rules for terms. The propositional sublogic is generated from the maximum and minimum, which are also the truth values. A powerful logical language arises when this base logic is combined with Peano's class term $\{x|f_x\}$; bound variables x and their occurrence in f_x are always written with lower case letters.

Verbalization:

x WITH $f_x := \{x|f_x\}$ — *class term*

Everything else is developed using explicit definitions in a way that they work as usual. In particular, this also includes the quantifiers, formalized according to Peano's definitions, with which the usual rules for quantifiers can be proved [3, p.75,140f]. The unique class with a given property can also be obtained, namely via the union of a one-element class [3, p.78]. In the philosophy of language, this operator is called definite description.

Definitions:

$0 := \neg\mathsf{I}$ — *empty class, false*
EQUAL $A := \{A\} := \{x|x{=}A\}$ — *single element class*
A IS A $B := A$ IS $B := A{\in}B := \{A\}{\cdot}B{\neq}0$ — *containment*
FOR ALL $x{:}f_x := \forall x{:}f_x := (\{x|f_x\}{=}\mathsf{I})$ — *universal quantifier*
EXISTS $x{:}f_x := \exists x{:}f_x := (\{x|f_x\}{\neq}0)$ — *existential quantifier*
ELEMENT OF ALL $A := \bigcap A := \{x|\forall y{:}(y{\in}A{\Rightarrow}x{\in}y)\}$ — *intersection*
ELEMENT OF AN $A := \bigcup A := \{x|\exists y{:}((y{\in}A){\cdot}(x{\in}y))\}$ — *union*
THE $A := \iota A := \bigcup A$ IF $\exists x{:}(A{=}\{x\})$ — *definite description*

The formal language used to prove that $\sqrt{2}$ is irrational is thus essentially the familiar mathematical language. The universal logic calculus therefore dispenses with any unnecessary formalism. All proofs are linear deductions as a sequence of terms between which the justifications are inserted as indices. The main justifications are equations and implications, some are rules of inference with inference operator $\vdash$. Each rule gets a name that signals its meaning. Rules used here and proved in the *Universallogik* with German names are named here in English, according to the following table.

Rules proved:		in [3]
$(A{\geq}B){\cdot}(B{\geq}A){\Rightarrow}(A{=}C)$	*antisymmetric*	p.158
automatic proof in a boolean Ring	*boolean logic*	p.128
$A{\cdot}\neg A = 0$	*contradiction*	p.125
$X{\in}A{\Rightarrow}X{\in}\mathrm{I}$	*existing*	p.132
$\exists x{:}(A{=}\{x\}) = \iota A{\in}A = \iota A{\in}\mathrm{I}$	*existing article*	p.144
$a{\Rightarrow}0 \vdash \neg a$	*falsification*	p.125
$p{\Rightarrow}f_X \vdash p{\Rightarrow}\forall x{:}f_x$	*general*	p.138
$(Y{\subseteq}\mathbb{N}){\cdot}(Y{\neq}0) \Rightarrow \exists a{\in}Y{:}\forall x{\in}Y{:}(x{\geq}a)$	*minimum exists*	p.167
$\exists x{:}(f_x{\cdot}p) = p{\cdot}\exists x{:}f_x$ for propositions f_X	*not quantified*	p.143
$(X{\in}A){\cdot}f_X{\Rightarrow}\exists x{\in}A{:}f_x$ for propositions f_X and p	*quantify*	p.138
$X{\in}\{x\|f_x\} = f_X{\cdot}(X{\in}\mathrm{I})$	*satisfied*	p.140
$X{\in}\{x{\in}A\|f_x\} = (X{\in}A){\cdot}f_X$	*satisfied*	p.143
$(X{\in}A){\cdot}\forall x{:}f_x{\Rightarrow}f_X$	*special*	p.138
$(X{\in}A){\cdot}\forall x{\in}A{:}f_x \Rightarrow f_X$	*special*	p.138
$A{\cdot}B{\subseteq}A$	*subset*	p.130

The complete calculus proof technique is described in a user-friendly manner on four pages in the univeral logic [3, pp.115-118]. It allows precise calculus proofs that fully correspond to the usual more informal proofs.

The following explanations of justifications may suffice here:

- Theorems, lemmas, axioms are used with their $_{name}$,
- definitions with $_{def}$, recognizable by the character eliminated or introduced.
- Hypotheses of the theorem or lemma proved are labeled with $_{hyp,\ hyp1,\ hyp2}$.
- Individual factors of proof terms are valid since the product is conjunction;
- this also applies for the domain of a quantifier.
- Underlined factors are referenced in sequential order with $_{U1,\ U2,\ \ldots}$.
- $_{-name}$ indicates the omission of a term or a factor.
- $_{+name}$ indicates the insertion of a factor.

Section 2 starts with a formalization of the original proof after Euclid with gcd. Section 3 recasts this proof with justifications in standard mathematical language. Section 4 gives a formalization of a shorter proof that avoids the notion of the gcd.

2 Proof with gcd according to Euclid

Quantities that Euclid used in the *Elements* for the proof are assumed together with their properties. They can be embedded order-isomorphically in the positive cone of a subgroup of the real numbers [2]. So it is permissible to take $\mathbb{R}^+$ as the maximum range of quantities. Of course, one can also extend the quantities with negative quantities to $\mathbb{R}$, as has become common in the 20th century.

Some calculation rules following from the Peano axioms are assumed to be known:

Calculation rules:	*arithmetic*
$2{\in}\mathbb{N},\ 2 \neq 1$	*number 2*
$A{\in}\mathbb{N} \Rightarrow A{\geq}1$	*minimal*
$(A{\in}\mathbb{N}){\cdot}(B{\in}\mathbb{N}){\Rightarrow}A{\bullet}B{\in}\mathbb{N}$	*in* $\mathbb{N}$
$(A{\in}\mathbb{N}){\cdot}(B{\in}\mathbb{N}){\Rightarrow}(A{\bullet}B{=}B{\bullet}A)$	*commutative*
$(A{\in}\mathbb{N}){\cdot}(B{\in}\mathbb{N}){\cdot}(C{\in}\mathbb{N}){\Rightarrow}((A{\bullet}B){\bullet}C = A{\bullet}(B{\bullet}C))$	*associative*
$A{\in}\mathbb{N}{\Rightarrow}(A{\bullet}1{=}A)$	*neutral*
$(A{\in}\mathbb{N}){\cdot}(B{\in}\mathbb{N}){\cdot}(C{\in}\mathbb{N}){\Rightarrow}((A{\bullet}B{=}A{\bullet}C) = (B{=}C))$	*reducible*

The same rules applied to real numbers, are denoted by $_{\textit{arithmetic in } \mathbb{R}}$. Squares and roots are defined in $\mathbb{R}$, also the natural numbers (according to Dedekind) so that the Peano axioms are provable [3, p.107], and also the positive rational numbers $\mathbb{Q}^+$ with extension to $\mathbb{Q}$.

Definitions:

$A^2 := A{\bullet}A$

THE ROOT OF $A := \sqrt{A} := \iota\{x|(x{\geq}0){\cdot}x^2{=}A\}$

NATURAL NUMBER $:= \mathbb{N} := \bigcap\{x|(1{\in}x){\cdot}\forall n{\in}x{:}(n+1{\in}x)\}$

NATURAL FRACTION $:= \mathbb{Q}^+ := \{x{\in}\mathbb{R}|\exists z, n{\in}\mathbb{N}{:}(x{\bullet}n{=}z)\}$

RATIONAL $:= \mathbb{Q} := \{x{\in}\mathbb{R}|(x{\in}\mathbb{Q}^+){\cdot}(x > 0){\vee}(-x{\in}\mathbb{Q}^+){\cdot}(0 > x){\vee}(x{=}0)\}$

Roots of natural numbers are of course in the area of positive quantities, to which the original proof is limited. Two lemmas show this.

Lemma 1: *square of the root*:

$\sqrt{A}{\in}X{\Rightarrow}(\sqrt{A}^2{=}A)$

Proof: $\sqrt{A}{\in}X$ $_{\textit{existing}}$ $\sqrt{A}{\in}\mathrm{I}$ $_{\textit{def existing article}}$ $\sqrt{A}{\in}\{x|(x{\geq}0){\cdot}(x^2{=}A)\}$ $_{\textit{satisfies}}$ $\sqrt{A}^2{=}A$ QED.

Lemma 2: *root of a natural number*:

$(\sqrt{A}\in\mathbb{Q})\cdot(A\in\mathbb{N})\Rightarrow\sqrt{A}\in\mathbb{Q}^{+}$

Proof: $(\sqrt{A}\in\mathbb{Q})\cdot(A\in\mathbb{N})$ $_{\textit{square of the root, minimal}}$ $(\sqrt{A}^{2}=A)\cdot(\underline{A\geq 1})$ $_{\textit{into U1}}$ $\sqrt{A}^{2}\geq 1$

$_{\textit{arithmetic in }\mathbb{R}}$ $\sqrt{A}\geq 1$ $_{\textit{arithmetic in }\mathbb{R}}$ $(\underline{\sqrt{A}>0})\cdot(\underline{0\not>\sqrt{A}})$ $_{\textit{def}}$ $\underline{\sqrt{A}\neq 0}$;

$_{\textit{hyp1}}$ $\sqrt{A}\in\mathbb{Q}$ $_{\textit{satisfies def}}$ $(\sqrt{A}\in\mathbb{Q}^{+})\cdot(\sqrt{A}>0)\vee(-\sqrt{A}\in\mathbb{Q}^{+})\cdot(0>\sqrt{A})\vee(\sqrt{A}=0)$

$_{\textit{-U2 +U3 +U4}}$ $(\sqrt{A}\in\mathbb{Q}^{+})\vee(-\sqrt{A}\in\mathbb{Q}^{+})\cdot(0\not>\sqrt{A})\cdot(0>\sqrt{A})\vee(\sqrt{A}\not>0)\cdot(\sqrt{A}=0)$

$_{\textit{boolean logic}}$ $\sqrt{A}\in\mathbb{Q}^{+}$ QED·

A few divisibility rules with associated definitions are also needed, because Euclid's proof is based on divisibility and the gcd.

Definitions:

A DIVIDES B $:=$ $A|B$ $:=$ $(A\in\mathbb{N})\cdot(B\in\mathbb{N})\cdot\exists n\in\mathbb{N}:(A\bullet n=B)$

A IS COPRIME WITH B $:=$ $A\perp B$ $:=$ $\forall x:((x|A)\cdot(x|B)\Rightarrow(x=1))$

GCD(A, B) $:=$ GCD(A, B) $:=$ $\iota\{x|(x|A)\cdot(x|B)\cdot\forall y:((y|A)\cdot(y|B)\Rightarrow x\geq y)$

Divisibility rules:

$(A\|B)\cdot(B\|A)\Rightarrow(A=B)$	*antisymmetric*
$(A\in\mathbb{N})\cdot(B\in\mathbb{N})\Rightarrow(A\|A\bullet B)\cdot(B\|A\bullet B)$	*extended numerator*
$(A\in\mathbb{N})\cdot(B\in\mathbb{N})\Rightarrow$GCD$(A, B)\in\mathbb{N}$	*gcd*
$(A\in\mathbb{N})\cdot(B\in\mathbb{N})\Rightarrow(GCD(A, B)\|A)\cdot(GCD(A, B)\|B)$	*common divisor*
$(X\|A)\cdot(X\|B)\Rightarrow X\|GCD(A, B)$	*subdivisor*
$A\in\mathbb{N}\Rightarrow(2\|A^{2} = 2\|A)$	*prime divisor*

Now the original propositions and proofs can be formulated exactly in the universal calculus.

Lemma 3: *relative prime*:

$(A\in\mathbb{N})\cdot(B\in\mathbb{N})\cdot(Z\in\mathbb{N})\cdot(N\in\mathbb{N})\cdot(GCD(A, B)\bullet Z=A)\cdot(GCD(A, B)\bullet N=B)\Rightarrow N\perp Z$

Proof: $hyp\cdot(X|Z)\cdot(X|N)\Rightarrow(X=1)$: $(X|Z)\cdot(X|N)$ $_{\textit{hyp1-4 +def}}$

$\underline{(A\in\mathbb{N})\cdot(B\in\mathbb{N})\cdot(Z\in\mathbb{N})\cdot(N\in\mathbb{N})\cdot(X\in\mathbb{N})}\cdot\exists a\in\mathbb{N}:(\underline{X\bullet a=Z})\cdot\exists b\in\mathbb{N}:(\underline{X\bullet b=N})$

$_{\textit{U2 into hyp5, U3 into hyp 6}}$ $\exists a\in\mathbb{N}:($GCD$(A, B)\bullet X\bullet a=A)\cdot\exists b\in\mathbb{N}:(GCD(A, B)\bullet X\bullet b=B)$

$_{\textit{def (U1 gcd in }\mathbb{N})}$ $($GCD$(A, B)\bullet X|A)\cdot($GCD$(A, B)\bullet X|B)$ $_{\textit{subdivisor +extended numerator}}$

$_{\textit{(U1 gcd in }\mathbb{N})}$ $($GCD$(A, B)\bullet X|($GCD$(A, B))\cdot($GCD$(A, B)|($GCD$(A, B)\bullet X)$

$_{\textit{antisymmetric}}$ GCD$(A, B)\bullet X=$GCD(A, B) $_{\textit{neutral}}$ GCD$(A, B)\bullet X=$GCD$(A, B)\bullet 1$

$_{\textit{reducible}}$ $X=1$; so: $_{\textit{hyp}}$ $(X|Z)\cdot(X|N)\Rightarrow(X=1)$ $_{\textit{general def}}$ $N\perp Z$ QED·

Lemma 4: *reduced fraction*:

$(A{\in}\mathbb{N})\cdot(B{\in}\mathbb{N}){\Rightarrow}\exists z, n{\in}\mathbb{N}{:}((n{\cdot}A{=}z{\cdot}B)\cdot(x{\perp}z))$

Proof: $(A{\in}\mathbb{N})\cdot(B{\in}\mathbb{N})$ $_{\textit{common divisor}}$ $(\mathrm{GCD}(A, B)|A)\cdot(\mathrm{GCD}(A, B)|B)$ $_{\textit{def>}}$
$\exists z{\in}\mathbb{N}{:}((\mathrm{GCD}(A, B){\cdot}z{=}A)\cdot\exists n{\in}\mathbb{N}{:}(\mathrm{GCD}(A, B){\cdot}n{=}B))$ $_{\textit{not quantified +commutative}}$
$\exists z, n{\in}\mathbb{N}{:}((\underline{\mathrm{GCD}(A, B){\cdot}z{=}A})\cdot(\underline{\mathrm{GCD}(A, B){\cdot}n{=}B})\cdot(\underline{B}{\cdot}A{=}\underline{A}{\cdot}B))$ $_{\textit{U1 into U4, U2 into U3}}$
$\exists z, n{\in}\mathbb{N}{:}(\mathrm{GCD}(A, B){\cdot}n{\cdot}A{=}\mathrm{GCD}(A, B){\cdot}z{\cdot}B)$ $_{\textit{reducible}}$ $\exists z, n{\in}\mathbb{N}{:}(n{\cdot}A{=}z{\cdot}B)$
$_{\textit{+relative prime (hyp U1+U2)}}$ $\exists z, n{\in}\mathbb{N}{:}((n{\cdot}A{=}z{\cdot}B)\cdot(x{\perp}z))$ QED$\cdot$

Theorem:

THE ROOT OF 2 IS NOT RATIONAL, $\sqrt{2}{\notin}\mathbb{Q}$

Proof: $_{\textit{falsification}}$: $\underline{\sqrt{2}{\in}\mathbb{Q}}$ $_{\textit{+number 2}}$ $(\sqrt{2}{\in}\mathbb{Q})\cdot(\underline{2{\in}\mathbb{N}})$ $_{\textit{root of a natural number}}$
$\sqrt{2}{\in}\mathbb{Q}^{+}$ $_{\textit{def satisfies}}$ $\exists a, b{\in}\mathbb{N}{:}(\underline{\sqrt{2}{\cdot}b{=}a})$ $_{\textit{reduced fraction-def}}$
$\exists a, b{\in}\mathbb{N}{:}\exists z, n{\in}\mathbb{N}{:}((n{\cdot}a{=}z{\cdot}b)\cdot\underline{\forall x{:}((x|z)\cdot(x|n){\Rightarrow}(x{=}1)))}$ $_{\textit{U3}}$
$\exists a, b{\in}\mathbb{N}{:}\exists z, n{\in}\mathbb{N}{:}(n{\cdot}\sqrt{2}{\cdot}b{=}z{\cdot}b)$ $_{\textit{arithmetic in } \mathbb{R} \textit{ not quantified}}$ $\exists z, n{\in}\mathbb{N}{:}(\sqrt{2}^{2}{\cdot}n^{2}{=}z^{2})$
$_{\textit{U1-square of the root,}}$ $\exists z, n{\in}\mathbb{N}{:}(\underline{2{\cdot}n^{2}{=}z^{2}})$ $_{\textit{quantify in } \mathbb{N}}$ $\exists z, n{\in}\mathbb{N}{:}\exists y{\in}\mathbb{N}{:}(2{\cdot}y = z^{2})$
$_{\textit{def}}$ $\exists z, n{\in}\mathbb{N}{:}(2|z^{2})$ $_{\textit{prime divisor}}$ $\exists z, n{\in}\mathbb{N}{:}(\underline{2|z})$ $_{\textit{def}}$ $\exists z, n{\in}\mathbb{N}{:}\exists x{\in}\mathbb{N}{:}(2{\cdot}x{=}z)$
$_{\textit{into U5}}$ $\exists z, n{\in}\mathbb{N}{:}\exists x{\in}\mathbb{N}{:}(2{\cdot}n^{2}{=}(2{\cdot}x)^{2})$ $_{\textit{arithmetic}}$ $\exists z, n{\in}\mathbb{N}{:}\exists x{\in}\mathbb{N}{:}(2{\cdot}x^{2}{=}n^{2})$
$_{\textit{as with U5: quantify in } \mathbb{N}\textit{, def, prime divisor, x not quantified}}$ $\exists z, n{\in}\mathbb{N}{:}(2|n)$ $_{\textit{+U6}}$
$\exists z, n{\in}\mathbb{N}{:}((2|z)\cdot(2|n))$ $_{\textit{U4-spezial}}$ $\exists z, n{\in}\mathbb{N}{:}(2{=}1)$ $_{\textit{not quantified +number 2}}$
$(2{=}1)\cdot(2{\neq}1)$ $_{\textit{contradiction}}$ 0 QED$\cdot$

3 The same proof with justifications in standard mathematical language

In order to make it clear that this proof in the calculus corresponds to the usual proof practice, it is now presented in a more freely annotated form. The justifications are integrated in the comment. As usual, we use numbered statements in place of the underlined factors in the calculus proof. Usually, leading existential quantifiers are hidden in the proof, although they should remain in exact logical language. The notation with lower case letters shows here that there are still bound variables. The hidden existential quantifiers are therefore restored when needed. A proof for more experienced readers would in addition skip some of the steps in the fully detailed proof.

Proof:

We assume

(1) $\sqrt{2}\in\mathbb{Q}$

and aim at a contradiction. Inserting *number2* we get

(2) $(\sqrt{2}\in\mathbb{Q})\cdot(2\in\mathbb{N})$

and using *root of a natural number*,

$\sqrt{2}\in\mathbb{Q}^+$

By definition of $\mathbb{Q}^+$,

(3) $\exists a, b\in\mathbb{N}:(\sqrt{2}\cdot b=a)$

Using *reduced fraction* with the definition of the coprime operator ⊥, we get

(4) $\exists a, b\in\mathbb{N}:\exists z, n\in\mathbb{N}:((n\cdot a=z\cdot b)\cdot\forall x:((x|z)\cdot(x|n)\Rightarrow(x=1)))$

By (3), hiding the quantifiers,

$n\cdot\sqrt{2}\cdot b=z\cdot b$

Arithmetic in $\mathbb{R}$ using (2) yields

$\sqrt{2}^2\cdot n^2=z^2$

By (1), the *square of the root* becomes 2, hence

(5) $2\cdot n^2=z^2$

Introduce the existential quantifier in $\mathbb{N}$

$\exists y\in\mathbb{N}:(2\cdot y = z^2)$

By definition of divisibility,

$2|z^2$

Using *prime divisor* we get

(6) $2|z$

and by definition,

$\exists x\in\mathbb{N}:(2\cdot x=z)$

We substitute this equation into (5), hiding the quantifier,

$2\cdot n^2=(2\cdot x)^2$

and use *Arithmetic* to get

$2\cdot x^2=n^2$

As before from equation (5) we find

$2|n$

With (6), restoring the hidden quantifiers we get

$\exists z, n\in\mathbb{N}:((2|z)\cdot(2|n))$

By *special* with the quantified factor in (4),

$\exists z, n\in\mathbb{N}:(2=1)$

Omitting quantifiers and inserting the inequality for number 2 we get

$(2=1)\cdot(2\neq 1)$

This *contradiction* proves the claim. QED.

4 Proof without gcd

Instead of working with reduced fractions, the existence of a fraction with the smallest numerator is sufficient and gives a much shorter proof using fewer rules and lemmas. From the divisibility rules one only needs the *prime divisor*, which says that the root of an even square number is also even.

Lemma 5: *smallest nominator*:

$\exists z,n\in\mathbb{N}:((A\cdot n=z)\Rightarrow\exists z,n\in\mathbb{N}:((A\cdot n=z)\cdot\forall x\in\mathbb{N}:(\exists y\in\mathbb{N}:(A\cdot y=x)\Rightarrow x\geq z))$

Proof: $_{\textit{abbreviaton}}\ \boldsymbol{M} := \{x\in\mathbb{N}|\exists y\in\mathbb{N}:(A\cdot y=x)\}$.
$_{\textit{hyp}}\ \exists z,n\in\mathbb{N}:(A\cdot n=z)\ _{\textit{def (bound variable renamed)}}\ \boldsymbol{M}\neq 0$
$_{\textit{+subset def of M}}\ (\boldsymbol{M}\subset\mathbb{N})\cdot(\boldsymbol{M}\neq 0)\ _{\textit{minimum}}\ \exists z:((z\in\boldsymbol{M})\cdot\forall x\in\boldsymbol{M}:(x\geq z))$
$_{\textit{satisfies abbreviaton}}\ \exists z,n\in\mathbb{N}:(A\cdot n=z):\forall x\in\mathbb{N}:(\exists y\in\mathbb{N}:(A\cdot y=x)\Rightarrow x\geq z)\ _{\text{QED}}\cdot$

Theorem:

THE ROOT OF 2 IS NOT RATIONAL, $\sqrt{2}\notin\mathbb{Q}$

Proof: $_{\textit{falsification}}$: $\underline{\sqrt{2}\in\mathbb{Q}}\ _{\textit{+number 2}}\ (\sqrt{2}\in\mathbb{Q})\cdot(\underline{2\in\mathbb{N}})\ _{\textit{root of a natural number}}$
$\sqrt{2}\in\mathbb{Q}^{+}\ _{\textit{def satisfies}}\ \exists n,z\in\mathbb{N}:(\underline{\sqrt{2}\cdot n=z})\ _{\textit{smallest nominator}}$
$\exists z,n\in\mathbb{N}:((\sqrt{2}\cdot n=z)\cdot\underline{\forall x\in\mathbb{N}:(\exists y\in\mathbb{N}:(\sqrt{2}\cdot y=x)\Rightarrow x\geq z))}\ _{\textit{arithmetic in }\mathbb{R}}$
$\exists z,n\in\mathbb{N}:(\sqrt{2}^{2}\cdot n^{2}=z^{2})\ _{\textit{U1-square of the root,}}\ \exists z,n\in\mathbb{N}:(\underline{2\cdot n^{2}=z^{2}})\ _{\textit{quantify in }\mathbb{N}}$
$\exists z,n\in\mathbb{N}:\exists y\in\mathbb{N}:(2\cdot y=z^{2})\ _{\textit{def}}\ \exists z,n\in\mathbb{N}:(2|z^{2})\ _{\textit{prime divisor}}\ \exists z,n\in\mathbb{N}:(2|z)\ _{\textit{def in }\mathbb{N}}$
$\exists z,n,x\in\mathbb{N}:(\underline{2\cdot x=z})\ _{\textit{into U5}}\ \exists z,n,x\in\mathbb{N}:(2\cdot n^{2}=(2\cdot x)^{2})\ _{\textit{arithmetic}}$
$\exists z,n,x\in\mathbb{N}:(2\cdot x^{2}=n^{2})\ _{\textit{as with U5: quantify in }\mathbb{N}\textit{, def, prime divisor}}\ \exists z,n,x\in\mathbb{N}:(2|n)\ _{\textit{def in }\mathbb{N}}$
$\exists z,n,x\in\mathbb{N}:\exists y\in\mathbb{N}:(\underline{2\cdot y=n})\ _{\textit{U6, U7 into U3}}\ \exists z,n,x\in\mathbb{N}:\exists y\in\mathbb{N}:(\sqrt{2}\cdot 2\cdot y=2\cdot x)$
$_{\textit{arithmetic in }\mathbb{R}}\ \exists z,n,x\in\mathbb{N}:\exists y\in\mathbb{N}:(\sqrt{2}\cdot y=x)\ _{\textit{spezial with U4}}\ \exists z,n,x\in\mathbb{N}:(x\geq z)\ _{\textit{U6}}$
$\exists z,n,x\in\mathbb{N}:(x\geq 2\cdot x)\ _{\textit{arithmetic not quantified +U2-minimal}}\ (1\geq 2)\cdot(2\geq 1)\ _{\textit{antisymmetric +number 2}}$
$(2=1)\cdot(2\neq 1)\ _{\textit{contradiction}}\ 0\ _{\text{QED}}\cdot$

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